Intermède : petite Intégrations par partie

Déterminer la primitive de $x \sin x$ grâce une Intégration par partie.

$\int x \sin x \, dx$ $\quad \int u'v + v'u = \int (uv)' \Leftrightarrow \int u'v + \int uv' = [uv]$

$\star \int u'v = [uv] - \int uv'$

$v = x \quad v' = 1$
$u = -\cos x \quad u' = \sin x$

$\star \int x \sin x \, dx = [-x \cos x] + \underbrace{\int \cos x \, dx}_{\sin x}$

$\int x \sin x = -x \cos x + \sin x$

Verif $\frac{d}{dx}(-x \cos x + \sin x)$
$= -\cos x + x \sin x + \cos x$
$= x \sin x$

$\int x^2 \cos x \, dx$

$v = x^2 \quad v' = 2x$
$u = \sin x \quad u' = \cos x$

$\star \int x^2 \cos x \, dx = [x^2 \sin x] - \int 2x \sin x \, dx$
$= x^2 \sin x - x \int x \sin x \, dx$
$= x^2 \sin x - 2(-x \cos x + \sin x)$
$= (x^2 - 2) \sin x + 2x \cos x$

Verif

$\frac{d}{dx}((x^2 - 2) \sin x + 2x \cos x)$
$= \cancel{2x \sin x} + (x^2 - 2) \cos x + 2 \cos x - \cancel{2x \sin x}$
$= x^2 \cos x$

Intégrations / Parties.

$$J = \int_1^e t^n \ln t \, dt$$

$u = \frac{t^{n+1}}{n+1}$ $v' \; 1/t$

$u' \; t^n$ $v \; \ln t$

$$\int u'v = [uv] - \int uv'$$
$$= \left[\frac{t^{n+1}}{n+1} \ln t\right]_1^e - \int \frac{t^{n+1}}{n+1} \frac{1}{t} dt$$
$$= \left[\frac{t^{n+1}}{n+1} \ln t\right]_1^e - \int \frac{t^n}{n+1} dt$$

$$\left[\frac{t^{n+1}}{n+1} \ln t\right]_1^e \Leftrightarrow \frac{e^{n+1}}{n+1} \ln e = \frac{e^{n+1}}{n+1}$$
$$\Leftrightarrow \frac{1^{n+1}}{n+1} \ln 1 = 0$$

$$\frac{e^{n+1}}{n+1} - \int_1^e \frac{t^n}{n+1} dt = \frac{e^{n+1}}{n+1} - \frac{1}{n+1}\left[\frac{t^{n+1}}{n+1}\right]_1^e$$

$$= \frac{e^{n+1}}{n+1} - \frac{1}{(n+1)^2}(e^{n+1} - 1) = \frac{(n+1)e^{n+1} - e^{n+1} + 1}{(n+1)^2}$$

$$J = \frac{n e^{n+1} + 1}{(n+1)^2}$$

$$\int \arctan t \, dt \qquad \int \frac{1}{t^2+1} dt$$

★ A Retenir Contrôle DE

$u' = 1$ $v = \arctan$

$u = t$ $v' = \frac{1}{t^2+1}$

$$\int u'v \, dt = [uv] - \int uv'$$
$$= t \arctan t - \int \frac{t}{t^2+1} dt$$

$$t \arctan t - \frac{1}{2} \ln(1+t^2)$$

$$L = \int (t-1) \sin t \, dt$$

$u' = \sin t$ $v = (t-1)$

$u = -\cos t$ $v' = 1$

$$\int v u' \, dt = [uv] - \int v' u$$
$$= [-\cos t \, (t-1)] - \int -\cos t$$
$$= -\cos t \, (t-1) + \int \cos t \, dt$$
$$= (1-t)\cos t + \int \cos t \, dt$$
$$= (1-t)\cos t + \sin t$$

$H = \int (t^2 - t + 1) e^{-t} \, dt$

$u' = 2t - 1 \qquad v = -e^{-t}$
$u = t^2 - t + 1 \qquad v' = e^{-t}$

$\int u v' = [u v] - \int u' v$

$\int (t^2 - t + 1) e^{-t} = (t^2 - t + 1) - e^{-t} - \int (2t - 1) - e^{-t}$

$u' = 2 \qquad v = e^{-t}$
$u = 2t - 1 \qquad v' = -e^{-t}$ $\Big\} \int 2e^{-t} dt$

$= (t^2 - t + 1) - e^{-t} - e^{-t} (2t - 1) - \int 2 e^{-t}$

$= (t^2 - t + 1) - e^{-t} - e^{-t} (2t - 1) - 2e^{-t}$

$= e^{-t} (-t^2 + t - 1 - 2t + 1 - 2)$

$H = e^{-t} (-t^2 - t - 2)$

$I = \int_0^1 \ln (1 + t^2) \, dt$

$u' = 1 \qquad v = \ln (1 + t^2)$
$u = t \qquad v' = 2t / 1 + t^2$

$= t \ln (1 + t^2) - 2 \int \frac{t^2}{1 + t^2}$

$u' = 1/1 + t^2 \qquad v = t^2$
$u = \ln (1 + t^2) \qquad v' = 2t$

$-2 \int_0^1 \left(1 - \frac{1}{t^2 + 1}\right) dt$

$I = \ln (2) - 2 \left[t - \arctan t \right]_0^1$

$= \ln (2) - 2 (1 - \arctan 1)$

$I = \ln 2 - 2 + 2 \frac{\pi}{4}$

$= \ln 2 - 2 + \frac{\pi}{2}$

Intégrale

a) Trouver la primitive de $ch x = \frac{e^x + e^{-x}}{2}$

b) Calcul par parties $\int (t+1) ch t \, dt$

a) $\frac{1}{2}\int (e^x + e^{-x}) \, dx = \frac{1}{2}(e^x - e^{-x}) = sh x$

$\int ch x \, dx = sh x$

b) $\int (t+1) \, ch t \, dt$

$u = (t+1) \quad v' = ch t = ch x$
$u' = (1) \quad v = sh t = sh x$

$\int (t+1) ch t \, dt = \left[(t+1) \frac{1}{2} e^x - e^{-x}\right] - \int \frac{e^x - e^{-x}}{2}$

$= \left[(t+1) \frac{1}{2} e^x - e^{-x}\right] - ch t$

$= (t+1) \, sh t - \int sh t \, dt.$
$= (t+1) \, sh t - ch t$

Exo

$\int_1^{e^\pi} \sin(\ln t) \, dt$

$u' = 1 \quad v = \sin(\ln t) \quad u = t \quad v' = \frac{\cos(\ln t)}{t}$

$N = \left[t \sin(\ln t)\right]_1^{e^\pi} - \int_1^{e^\pi} \cancel{t} \frac{\cos(\ln t)}{\cancel{t}} \, dt$

$= -\int_1^{e^\pi} \cos(\ln t) \, dt$

$u' = 1 \quad v = \cos(\ln t) \quad u = t \quad v' = \frac{-\sin(\ln t)}{t}$

$\left[t \cos(\ln t)\right]_1^{e^\pi} - \int \cancel{t} - \frac{\sin(\ln t)}{\cancel{t}}$

$\left[t \cos(\ln t)\right]_1^{e^\pi} + \int_1^{e^\pi} \sin(\ln t)$

$\int_1^{e^\pi} \cos(\ln t) = \left[t \cos(\ln t)\right]_1^{e^\pi} + \int_1^{e^\pi} \sin(\ln t) \, dt$

$N = -\left[t \cos(\ln t)\right]_1^{e^\pi} - N \iff N = \frac{-1}{2}\left[t \cos(\ln t)\right]_1^{e^\pi}$

$N = -\frac{1}{2}(-e^\pi - 1) = \frac{e^\pi + 1}{2}$

Exo Une triple.

a) calculer $\int t \sin t \, dt$

b) En déduire $A = \int t \sin^3 t \, dt$

$\int t \sin t \, dt$.

$u = t \quad v' = \sin t$
$u' = 1 \quad v = -\cos t$

$$\int t \sin dt = -t \cos t - \int -\cos t \, dt$$
$$= -t \cos t + \int \cos t \, dt$$
$$= -t \cos t + \sin t$$

$$A = \int t \sin^3 t \, dt = \int t \sin t \, (1 - \cos^2 t) dt$$
$$= \underbrace{\int t \sin t \, dt}_{\text{FAIT}} - \underbrace{\int t \sin t \cos^2 t \, dt}_{\text{Ipp } B}$$

$$B = \int t \sin t \cos^2 t \, dt$$

$\frac{d}{dt}\left(-\frac{1}{3}\cos^3 t\right) = \cos^2 t \, (\sin t)$ forme $\propto u^2 u'$

$u' = \cos^2 t \sin t \quad u = -\frac{1}{3}\cos^3 t \quad v = t \quad v' = 1$

$$\left[-\frac{1}{3}\cos^3 t \;\; t\right] - \int -\frac{1}{3}\cos^3 t$$

$$= -\frac{1}{3}\cos^3 t \;\; t + \frac{1}{3}\int \cos^3 t \qquad \cos^3 t = \cos t \underbrace{\cos^2 t}_{(1-\sin^2 t)}$$

$$\int \cos^3 t \, dt = \int \cos t \, (1 - \sin^2 t) dt$$
$$= \int \cos t \, dt - \int \cos t \sin^2 t \, dt$$
$$\int \cos^3 t \, dt = \sin t - \frac{1}{3}\sin^3 t$$

$$\int t \sin^3 t \, dt = -t \cos t + \sin t - \left(-\frac{t}{3}\cos^3 t + \frac{1}{3}\left(\sin t - \frac{1}{3}\sin^3 t\right)\right)$$
$$= -t \cos t + \cancel{\sin t} + \frac{t}{3}\cos^3 t - \cancel{\frac{1}{3}\sin t} + \frac{2}{3}\sin^3 t + \frac{1}{9}\sin^3 t$$
$$= -t \cos t + \frac{2}{3}\sin t + \frac{t}{3}\cos^3 t + \frac{1}{9}\sin^3 t + C$$

Intégration / Partie

Intégrale de Wallis

$n \in \mathbb{N}$, $I_n = \int_0^{\pi/2} \sin^n x \, dx$

1) Montrons pour $n \geqslant 2$, que $I_n = \frac{n-1}{n} I_{n-2}$

On fera une intégration par parties.

$\int u'v \, dx = uv \int uv' \, dx$

$(f(g(x)))' = f'(g(x)) \cdot g'(x)$

$I_n = \int_0^{\pi/2} \sin^{n-1} x \sin x \, dx$

$(\sin x)^{n-1}$

$u' = \sin x$; $v = \sin^{n-1} x$; $u = -\cos x$; $v' = n-1 \sin^{n-2} x \ (\cos x)$

$I_n = \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} (-\cos x)(n-1)(\sin^{n-2} x)(\cos x) \, dx$

$I_n = (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$

On sait (trigo hyperbolique) $\cos^2 x = 1 - \sin^2 x$

$I_n = (n-1) \int_0^{\pi/2} \sin^{n-2} x \, (1 - \sin^2 x) \, dx = n-1$

$= (n-1) \left(\int_0^{\pi/2} \sin^{n-2} x \, dx - \int_0^{\frac{\pi}{2}} \sin^n x \, dx \right)$

$= (n-1)(I_{n-2} - I_n)$

$I_n (1 + n - 1) = (n-1) I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2}$

Pr Vendredi Montrer que $\forall p \in \mathbb{N}$

$I_{2p} = \frac{(2p)!}{2^{2p}(p!)^2} \frac{\pi}{2}$