TD 4

- calcul de primitives
- Transformation.
- IPP | - changement de variable.

$f(x) = \frac{x^2+x+1}{x}$
$= \frac{x^2}{x} + \frac{x}{x} + \frac{1}{x}$
$= x+1+\frac{1}{x}$

Exercice n°3

$(u^n)' = nu'u^{n-1}$ $(\ln|u|)' = \frac{u'}{u}$ $(e^u)' = u'e^u$

1- $\int_0^1 te^{t^2}\,dt$

$(e^u)' = u'e^u$
$u(t) = t^2$
$u'(t) = 2t$

$= \frac{1}{2}\int_0^1 \underbrace{2t\,e^{t^2}}\,dt$

$= \frac{1}{2}\left[e^{t^2}\right]_0^1 = \frac{e-1}{2}$

2-) $\int_0^1 t^2(t^3+2)^5\,dt$

$(u^n)' = \underline{nu'u^{n-1}}$
$u(t) = t^3+2$
$u'(t) = 3t^2$
$n-1 = 5$
$n = 6$

$= \frac{1}{18}\int_0^1 6\times 3t^2(t^3+5)^5\,dt$

$= \frac{1}{18}\left[(t^3+2)^6\right]_0^1$

$= \frac{3^6-2^6}{18}$

$\int_0^1 t(t^3+2)^5\,dt$

$= \int_0^1 \frac{1}{18}(8t^2(t^2+2)^5\,dt$

pas possible.
⚠ prob de puissance et dénominateur

1er degré

6ème degré

$$3- \int_0^1 \frac{t+1}{(t^2+2t+3)^5}\,dt = \int_0^1 (t+1)(t^2+2t+3)^{-3}\,dt$$

$(u^n)' = nu'u^{n-1}$

$u(t) = t^2+2t+3$

$u'(t) = 2t+2$

$n-1 = -3$

$n = -2$

$$= -\frac{1}{4}\int_0^1 -2\times 2(t+1)(t^2+2t+3)^{n-1}\,dt$$

$$= -\frac{1}{4}\left[(t^2+2t+3)^{-2}\right]_0^1$$

$$= -\frac{1}{4}\left(\frac{1}{36}-\frac{1}{9}\right) = \boxed{\frac{1}{48}}$$

$$4-)\ \int_0^1 \frac{t+5}{t^2-2t-3}\,dt$$

$(\ln|u|)' = \frac{u'}{u}$

$u(t) = t^2-2t-3$

$u'(t) = 2t-2$

$= 2(t-1)$

$$\times\ = \int_0^1 \frac{t+5}{2(t+1)} \times \frac{2(t-1)}{t^2-2t-3}\,dt$$

$$\frac{1}{6} = -\frac{1}{3}+\frac{1}{2}$$

$$t^2-2t-3 = (t+1)(t-3)$$

il existe 2 réels A et B tels que

$$\frac{t+5}{(t+1)(t-3)} = \frac{A}{(t+1)} + \frac{B}{(t-3)}$$

$\times t+1 \quad \frac{t+5}{t-3} = A + \frac{B(t+1)}{t-3}$

$t=-1 \quad \boxed{A=-1}$

$\times t-3;\ t=3 \quad \boxed{B=2}$

$$\int_0^1 \frac{t+5}{t^2-2t-3}\,dt = \int_0^1 -\frac{1}{t+1} + 2\times\frac{1}{t-3}\,dt$$

$$= \left[-\ln|t+1| + 2\ln|t-3|\right]_0^1$$

$$= -\ln 2 + 2\ln 2 - 2\ln 3$$

$$= \underline{\ln 2 - 2\ln 3}$$

<u>Exercice n°4</u>

$$2-\int_{-1}^{0} \frac{2}{(x-1)^2(x^2+1)}\,dt$$

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\frac{2}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$\times\, x+1$; $x-1$ $\quad \boxed{C=1}$

$\times\, x^2$; $x=0$ $\quad \boxed{B=1}$

$x=1$

$$A+B+\frac{C}{2} = \frac{1}{2}$$

$$A+1+\cancel{\frac{1}{2}} = \cancel{\frac{1}{2}}$$

$$\boxed{A=-1}$$

$\times\, x$

$$\underbrace{A+\frac{B}{x}+\frac{Cx}{x+1}}_{x\to+\infty} = \frac{1}{x(x+1)}$$

$$A+C=0$$

$$A=-C=\boxed{-1}$$

$$\int_1^2 \frac{dx}{x^2(x+1)} = \int_1^0 \left(-\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x+1}\right)dx$$

$$= \left[-\ln x - \frac{1}{x} + \ln|x+1|\right]_1^2$$

$$= -\ln 2 - \frac{1}{2} + \ln 3 + 1 - \ln 2$$

$$= \ln 3 - 2\ln 2 + \frac{1}{2}$$

$$\frac{2}{(x-1)^2(x^2+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$\times (x-1)^2$, $x = 1$ $\boxed{B = 1}$

$\times (x^2+1)$ $x = i$ $Ci+D = \frac{2}{(i-1)^2} = \frac{2}{-2i} = i$

$\boxed{C=1}$ et $\boxed{D=0}$

$x=0$ $2 = -A + B + D$

$A = 1 - 2 = \boxed{-1}$

$$\int_{-1}^{0} \frac{2}{(x-1)^2(x^2+1)} dx = \int_{-1}^{0} -\frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{x}{x^2+1} dx$$

$$= \left[-\ln|x-1| - \frac{1}{x-1} + \frac{1}{2}\ln|x^2+1|\right]_{-1}^{0}$$

$$= \frac{1+\ln 2}{2}$$

$\int uv' = [uv] - \int u'v$

Exercice n°5

$(e^u)' = u'e^u$

On essaye de se débarrasser du (t^2+1) 2 IPP

1-) $\int_0^1 (t^2+1)e^{-3t} dt$.

$u(t) = t^2+1$ | $u'(t) = 2t$
$v'(t) = e^{-3t}$ | $v(t) = -\frac{1}{3}e^{-3t}$

$$\int_0^1 (t^2+1)e^{-3t} dt = \left[-\frac{1}{3}e^{-3t}(t^2+1)\right]_0^1 + \frac{2}{3}\int_0^1 te^{-3t} dt =$$

$u(t) = t$ | $u'(t) = 1$
$v'(t) = e^{-3t}$ | $v(t) = -\frac{1}{3}e^{-3t}$

$$\left[-\frac{1}{3}e^{-3t}(t^2+1)\right]_0^1 + \frac{2}{3}\int_0^1 te^{-3t}\,dt$$

$$= \left[-\frac{1}{3}e^{-3t}(t^2+1)\right]_0^1 + \frac{2}{3}\left\{\left[-\frac{1}{3}e^{-3t}\times t\right]_0^1 + \frac{1}{3}\int_0^1 e^{-3t}\,dt\right\}$$

$$= \left[-\frac{1}{3}e^{-3t}(t^2+1)\right]_0^1 + \frac{2}{3}\left[-\frac{1}{3}e^{-3t}t\right]_0^1 + \frac{2}{9}\left[-\frac{1}{3}e^{-3t}\right]_0^1$$

$$= \left[-\frac{1}{3}e^{-3t}\left(t^2+1+\frac{2}{3}t+\frac{2}{9}\right)\right]_0^1 = -\frac{e^{-3}}{3}\left(1+1+\frac{2}{3}+\frac{2}{9}\right)+\frac{1}{3}\left(1+\frac{2}{9}\right)$$

$$= \underline{\frac{11-26e^{-3}}{27}}$$

2-)

IPP $\int_1^e 3t\,\mathrm{Ln}t\,dt$

$$u(t) = \mathrm{Ln}t \quad \Big| \quad u'(t) = \frac{1}{t}$$
$$v'(t) = 3t \quad \Big| \quad v(t) = \frac{3t^2}{2}$$

$$\int_1^e 3t\,\mathrm{Ln}t\,dt = \left[\frac{3t^2}{2}\ln t\right]_1^e - \frac{3}{2}\int_1^e \frac{t^2}{t}\,dt$$

$$= \left[\frac{3}{2}t^2\ln t\right]_1^e - \frac{3}{2}\left[\frac{t^2}{2}\right]_1^e$$

$$= \left[\frac{3}{2}t^2\left(\mathrm{Ln}t - \frac{1}{2}\right)\right]_1^e = \boxed{\frac{3}{4}e^2 + \frac{3}{4}}$$

2IPP 3-) $\int_0^{\pi/4} t^2\cos 2t\,dt$ $\qquad (\sin u)' = u'\cos u$

$$u(t) = t^2 \quad \Big| \quad u'(t) = 2t$$
$$v'(t) = \cos 2t \quad \Big| \quad v(t) = \frac{1}{2}\sin 2t$$

$$\int_0^{\pi/4} t^2\cos 2t\,dt = \left[\frac{1}{2}t^2\sin 2t\right]_0^{\pi/4} - \int_0^{\pi/4} t\sin 2t\,dt$$

$$= \left[\frac{1}{2}t^2\sin 2t\right]_0^{\pi/4} - \left[\frac{t}{2}\cos 2t\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4}\cos 2t\,dt$$

$$= \left[\frac{1}{2}t^2\sin 2t\right]_0^{\pi/4} + \left[\frac{t}{2}\cos 2t\right]_0^{\pi/4} - \frac{1}{2}\left[\frac{1}{2}\sin 2t\right]_0^{\pi/4}$$

$$= \left[\left(\frac{1}{2}t^2 - \frac{1}{4}\right)\sin 2t + \frac{t}{2}\cos 2t\right]_0^{\frac{\pi}{4}}$$

$$= \boxed{\frac{\pi^2}{32} - \frac{1}{4}}$$

4-) $\int_0^1 \cos 2\pi t\, e^t\, dt$

$u(t) = \cos 2\pi t \quad | \quad u'(t) = 2\pi \sin 2\pi t$

$v'(t) = e^t \quad | \quad v(t) = e^t$

$$= \left[e^t \cos 2\pi t\right]_0^1 + 2\pi \int_0^1 \sin(2\pi t)\, e^t\, dt$$

$$= \left[e^t \cos 2\pi t\right]_0^1 + 2\pi\left[\sin(2\pi t)\, e^t\right]_0^1 - 4\pi^2 \int_0^1 \cos(2\pi t)\, e^t\, dt$$

$$= \left[e^t\left(\cos 2\pi t + 2\pi \sin 2\pi t\right)\right]_0^1 - 4\pi^2 I$$

$$= (1+4\pi^2) I = e - 1 \Rightarrow I = \frac{e-1}{1+4\pi^2}$$

Exercice n°6

- Décompostion en élément simple (methode de tranformation) -> efficace pour fraction

1- $\int_0^1 \ln(1+t^2)\, dt$

$u(t) = \ln(1+t^2) \quad | \quad u'(t) = \frac{2t}{1+t^2}$

$v'(t) = 1 \quad | \quad v(t) = t$

$$\int_0^1 \ln(1+t^2)\, dt = \left[t \ln(1+t^2)\right]_0^1 - 2\int_0^1 \frac{t^2}{1+t^2}$$

fraction rationelle

$$\begin{array}{r|l} t^2 & t^2+1 \\ \hline -(t^2+1) & 1 \\ \hline -1 & \end{array} \qquad \frac{t^2}{t^2+1} = 1 - \frac{1}{1+t^2}$$

$$= \left[t \ln(1+t^2)\right]_0^1 - 2\int_0^1 \left(1 - \frac{1}{1+t^2}\right) dt$$

$$= \left[t \ln(1+t^2)\right]_0^1 - 2\left[t - \arctan t\right]_0^1 =$$

$= \left[t \ln(1+t^2) - 2t + 2\arctan t \right]_0^1 = \ln 2 - 2 + 2 \times \frac{\pi}{4} =$

$\boxed{\ln 2 - 2 + \frac{\pi}{2}}$

Autre méthode.

$\int_0^1 \frac{t^2}{1+t^2} \, dt$

Posons $t = \tan\theta$, $\theta = \arctan t$ | $\frac{dt}{d\theta} = (1+\tan^2\theta)$, $dt = (1+\tan^2\theta) \, d\theta$

$$\int_0^1 \frac{t^2}{1+t^2} \, dt = \int_0^{\pi/4} \frac{\tan^2\theta}{\cancel{1+\tan^2\theta}} \cancel{(1+\tan^2\theta)} \, d\theta$$

$$= \int_0^{\pi/4} \tan^2\theta \, d\theta = \int_0^{\pi/4} 1+\tan^2\theta - 1 \, d\theta$$

$$= \left[\tan\theta - \theta \right]_0^{\pi/4}$$

$(u^2)' = 2u'u$

2-) $\int_1^e \frac{\ln t}{t} \, dt$

$I = \int_1^e \frac{1}{t} \times \ln t \, dt$

$= \frac{1}{2} \int_1^e 2 \times \frac{1}{t} \ln t \, dt$

$= \frac{1}{2} \left[(\ln t)^2 \right]_1^e$

$= \frac{1}{2}$

$u(t) = \ln t \quad u'(t) = \frac{1}{t}$

$v'(t) = \frac{1}{t} \quad v(t) = \ln t$

$I = \left[(\ln t)^2 \right]_1^e - \int_1^e \frac{\ln t}{t} \, dt$

$I = 1 - I$

$2I = 1$

$I = \frac{1}{2}$

changement de variable

Posons $x = \ln t$

$t = e^x$

$\frac{dt}{dx} = e^x$

$dt = e^x dx$

$I = \int_0^1 \frac{x}{e^x} e^x \, dx$

$= [\frac{1}{2} x^2]_0^1 = \frac{1}{2}$

3) $\int_0^{\pi/4} \sin(3t) \cos(2t) \, dt = \frac{1}{2} \int_0^{\pi/4} (\sin 5t + \sin t) dt = *$

① double IPP

② Formule d'euler -> Linéarisation

$\sin(3t)\cos(2t) = \frac{e^{3it} - e^{-3it}}{2i} \times \frac{e^{2it} + e^{-2it}}{2}$

$= \frac{e^{5it} + e^{it} - e^{-it} - e^{-5it}}{4i}$

$= \frac{e^{5it} - e^{-5it} + e^{it} - e^{-it}}{4i} = \frac{2i \sin 5t + 2i \sin t}{4i}$

$= \frac{1}{2} (\sin 5t + \sin t)$

$* = \frac{1}{2} [-\frac{\cos 5t}{5} - \cos t]_0^{\pi/4}$

$= \frac{1}{2} \left(\frac{\sqrt{2}}{10} - \frac{\sqrt{2}}{2} + \frac{1}{5} + 1 \right) = \frac{1}{2} \left(-\frac{2\sqrt{2}}{5} + \frac{6}{5} \right) = \frac{3 - \sqrt{2}}{5}$

Exercice n°6

$$\int_0^1 \cos(2\pi t)\sin^2(\pi t)\,dt$$

$$\cos(2\pi t)\sin^2(\pi t) = \frac{e^{2i\pi t}+e^{-2i\pi t}}{2} \times \left(\frac{e^{i\pi t}-e^{-i\pi t}}{2i}\right)^2$$

$$= \frac{e^{2i\pi t}+e^{-2i\pi t}}{2} \times \frac{e^{2i\pi t}+e^{-2i\pi t}-2}{-4}$$

$$= \frac{e^{4i\pi t}+e^{-4i\pi t}+2-2\left(e^{2i\pi t}+e^{-2i\pi t}\right)}{-8}$$

$$= \frac{2\cos 4\pi t+2-4\cos 2\pi t}{-8}$$

$$= -\frac{1}{4}\left(\cos 4\pi t+1-2\cos 2\pi t\right)$$

$$I = -\frac{1}{4}\int_0^1 \left(\cos 4\pi t+1-2\cos 2\pi t\right)dt$$

$$= -\frac{1}{4}\left[\frac{\sin 4\pi t}{4\pi} \times t - \frac{\sin 2\pi t}{\pi}\right]_0^1$$

$$= -\frac{1}{4}$$

Exercice n°7

1) $I = \int_0^{\sqrt{\pi}} t\sin(t^2)\,dt$ $\quad u(t) = t^2$

$u'(t) = 2t$

1ère =

$$I = -\frac{1}{2}\int_0^{\sqrt{\pi}} -2t\sin t^2\,dt = -\frac{1}{2}\left[\cos t^2\right]_0^{\sqrt{\pi}}$$

$$= -\frac{1}{2}(-1-1) = \boxed{1}$$

2ème Posons $x = t^2$ $\quad t = \sqrt{x}$ $\quad \frac{dt}{dx} = \frac{1}{2\sqrt{x}}$

$$I = \int_0^{\sqrt{\pi}} t \sin(t^2)\, dt = \int_0^{\pi} \cancel{\sqrt{x}} \sin x \frac{1}{2\cancel{\sqrt{x}}}\, dx = \frac{1}{2}$$

$$\int_0^{\pi} \sin x\, dx = \frac{1}{2} [-\cos x]_0^{\pi} = 1$$

$$\int_1^e \frac{\ln t}{t(\ln^2 t + 1)}\, dt \qquad x = \ln t$$

$$\int_0^1 \frac{\sqrt{t+2}}{t+1}\, dt \qquad x = \sqrt{t+2}$$

$$\int_0^1 \frac{\arctan x}{x^2}\, dx \qquad \text{IPP } u(x) = \arctan x \quad v'(x) = \frac{1}{x^2}$$

2) $\int_1^e \frac{\ln t}{t(\ln^2 t + 1)}\, dt = \int_0^1 \frac{x}{\cancel{e^x}} \cancel{e^x}\, dx$

$x = \ln t$

$t = e^x$

$\frac{dt}{dx} = e^x$ donc $dt = e^x dx$

$$= \frac{1}{2} \int_0^1 \frac{2x}{1+x^2}\, dx$$

$$= \frac{1}{2} \left[\ln(1+x^2)\right]_0^1$$

$$= \frac{\ln 2}{2}$$

$$\int_0^1 \frac{\sqrt{t+2}}{t+1}\,dt = \int_{\sqrt{2}}^{\sqrt{3}} \frac{x}{x^2-2+1} \times 2x\,dx$$

$x = \sqrt{t+2}$

$t = x^2 - 2$

$\frac{dt}{dx} = 2x$

$dt = 2x\,dx$

$$(*) = 2\int_{\sqrt{2}}^{\sqrt{3}} \frac{x^2}{x^2-1}\,dx$$

$$\begin{array}{r|l} x^2 & x^2-1 \\ \hline -(x^2-1) & 1 \\ 1 & \end{array}$$

$x^2 = (x^2-1)\times 1 + 1$

$$\frac{x^2}{x^2-1} = 1 + \frac{1}{x^2+1}$$

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)}$$

$$= \frac{a}{(x-1)} + \frac{b}{(x+1)}$$

$* x-1 : x = 1$, $a = \frac{1}{2}$ | $\times x+1, x = -1$, $b = -\frac{1}{2}$

$$(*)\ 2\int_{\sqrt{2}}^{\sqrt{3}} \left(1 + \frac{1}{2} * \frac{1}{x-1} - \frac{1}{2} \times \frac{1}{x+1}\right) dx$$

$$= 2\left[x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln(x+1)\right]_{\sqrt{2}}^{\sqrt{3}}$$

$$= \underline{2\left(\left[\sqrt{3} - \sqrt{2} + \frac{1}{2}\left(\ln(\sqrt{3}-1) - \ln(\sqrt{2}-1) - \ln(\sqrt{3}+1) + \ln(\sqrt{2}+1)\right.\right.\right.}$$

$$= 2(\sqrt{3} - \sqrt{2}) + \ln(8\sqrt{2} - 6 - 2\sqrt{3})$$

4-) $\int_{1/2}^{1} \frac{\arctan x}{x^2}\, dx$

$u(x) = \arctan x \qquad u'(x) = \frac{1}{1+x^2}$

$v'(x) = \frac{1}{x^2} \qquad v(x) = -\frac{1}{x}$

$I = \left[-\frac{\arctan x}{x} \right]_{\frac{1}{2}}^{1} + \int_{\frac{1}{2}}^{1} \frac{1}{x(1+x^2)}\, dx$

$\frac{1}{x(1+x^2)} = \frac{a}{x} + \frac{bx+c}{1+x^2}$

* x, $x = 0$ $\boxed{a = 1}$
* $1+x^2$, $x = i$ $\quad bi + c = \frac{1}{i} = -i$

$\boxed{c = 0 \text{ et } b = -1}$

$I = \left[-\frac{\arctan x}{x} \right]_{\frac{1}{2}}^{1} + \int_{\frac{1}{2}}^{1} \frac{1}{x} - \frac{x}{1+x^2}\, dx$

$= \left[-\frac{\arctan x}{x} + \ln x - \frac{1}{2} \ln(1+x^2) \right]_{\frac{1}{2}}^{1}$

$= -\frac{\pi}{4} - \frac{1}{2}\ln 2 + 2\arctan\frac{1}{2} + \ln 2 + \frac{1}{2}\ln\frac{5}{4}$

$= 2\arctan\frac{1}{2} - \frac{\pi}{4} + \frac{1}{2}\ln\frac{5}{2}$

5-) $\int_{0}^{1} \frac{x}{(x^2+x+1)^2}\, dx$

$= \int_0^1 \frac{x}{\left[\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}\right]^2} dx = \int_0^1 \frac{x}{\left[\frac{3}{4}\left(1+\frac{4}{3}\left(x+\frac{1}{2}\right)^2\right)\right]^2} dx$

$= \frac{16}{9} \int_0^1 \frac{x}{\left[1+\frac{4}{3}\left(x+\frac{1}{2}\right)^2\right]^2}$

$$\int \frac{x}{(x^2+1)^2}\,dx$$

$$t^2 = \frac{4}{3}\left(x+\frac{1}{2}\right)^2$$

$$= \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)$$

$$x = \frac{\sqrt{3}}{2}t - \frac{1}{2}$$

$$dx = \frac{\sqrt{3}}{2}\,dt$$

$$I = \frac{16}{9}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}t - \frac{1}{2}}{(1+t^2)^2} \times \frac{\sqrt{3}}{2}\,dt$$

$$I = \frac{4\sqrt{3}}{9}\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{\sqrt{3}t - 1}{(1+t^2)^2}\,dt$$

$$I = \frac{4}{3}\times\frac{1}{2}\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2t}{(1+t^2)^2}\,dt - \frac{4\sqrt{3}}{9}\int_{1\sqrt{3}}^{\sqrt{3}} \frac{1}{(1+t^2)^2}\,dt$$

$$I = \frac{2}{3}\left[-\frac{1}{1+t^2}\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} - \frac{4\sqrt{3}}{9}\,J$$

$$J = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1+t^2-t^2}{(1+t^2)^2}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1+t^2}{(1+t^2)^2}\,dt - \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{t^2}{(1+t^2)^2}\,dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+t^2}\,dt - \int_{1/\sqrt{3}}^{3} t \times \frac{t}{(1+t^2)^2}\,dt$$

$u(t) = t \qquad u'(t) = 1$

$v'(t) = \frac{t}{(1+t^2)^2} \qquad v(t) = \frac{1}{2} \times \frac{1}{1+t^2}$