

## TD Polynomes

A - Polynomes à coefficients réels

$$A = B \cdot Q + R$$

$$d^o R < d^o B$$

$$A = (x \sin \varphi + \cos \varphi)^4 \quad 1-) \quad (x \sin \varphi + \cos \varphi)^4 = (x^2 + 1) \times Q + R$$

$$B = (x^2 + 1) \quad R < 2$$

$$R = \alpha x + b \Rightarrow \alpha, b \in \mathbb{R}$$

$$(x \sin \varphi + \cos \varphi)^4 = (x^2 + 1) \times Q + (\alpha x + b)$$

$$x = 0 \quad (\cos \varphi)^4 = Q(0) + b$$

$$x = i \quad (i \sin \varphi + \cos \varphi)^4 = 0 \cdot Q(i) + \alpha i^2 + b$$

$$\cos \varphi + i \sin \varphi = b + i\alpha$$

$\alpha$  et  $b$  réels

$$\alpha = \sin 4\varphi$$

$$b = \cos 4\varphi$$

$$A \in \mathbb{R}[x] \Rightarrow Q \in \mathbb{R}[x]$$

$$B \in \mathbb{R}[x] \quad R \in \mathbb{R}[x]$$

3 -)

$$x^{3m} + x^{3n+2} + x^{3p+2} = (x^2 + x + 1) \times Q + R \quad d^o R < d^o B = 2$$

$$\Rightarrow R = 0$$

$$R = \alpha x + b$$

$$x^2 + x + 1 = 0$$

$$j_1 = x_1 = \frac{-1 + i\sqrt{3}}{2}$$

$$j_2 = x_2 = \frac{-1 - i\sqrt{3}}{2}$$

$$1 + x + x^2 = \frac{1 - x^3}{1 - x} \quad x \neq 1$$

$$j = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$$

$$j^2 = \bar{j} \cos \frac{2\pi}{3} - i \sin \frac{2\pi}{3}$$

$$1+j+j^2=0$$

$j$  racine cubique de l'unité.

$$\hookrightarrow j^3=1 \quad j^{3m}=(j^3)^m=1$$

$$j^{3n+2}=j^{3m}+j^2=j$$

$$j^{3p+2}=-j^2$$

$$\text{d'où } x^{3p+2} + x^{3n+2} + x^{3m} = 0$$

$$(j^2)^{3m}=j^{6m}=1$$

$$(j^2)^{3n+2}=j^2$$

$$(j^2)^{3p+2}=j^4=j^{3+1}=j$$

( P coefficients réels

P admet une racine complexe  $z_1$

$$\Rightarrow \bar{z}_1 \text{ et aussi racine } P(z_1)=0 \quad \bar{P}(z_1)=0 \quad P(\bar{z}_1)=0$$

$j$  racine de A

$\bar{j}$  racine de A  $\Rightarrow (x-j)(x-\bar{j})$   
divise A.

donc  $x^2+x+1$  divise A

$$\forall (m, n, p) \in \mathbb{N}^3$$

$$2-) -x^3 + 3x^2 + 2x = (-x^3 + 2x + 1) Q + x^{n+4} R$$

$$\begin{array}{r}
 1 + 2x - x^3 \\
 \hline
 2x - x^2 + x^3 - x^5 \\
 \hline
 0 - x^2 - x^3 + 2x^4 \\
 \hline
 -(-x^2 - 2x^3 + x^5) \\
 \hline
 0 + x^3 + 2x^4 - x^5 \\
 \hline
 -(x^3 + 2x^4) \\
 \hline
 -x^5
 \end{array}$$

$$2x^3 + 3x^2 - x^3 = (1 + 2x - x^3)(2x - x^2 + x^3) + x^5(-1 + x)$$

### Exercice n°6

$$\begin{array}{r}
 x^{99} + 1 \\
 \hline
 -(x^{99} + x^{54} + 1) \\
 \hline
 0 - x^{54} + 1 \\
 \hline
 -(x^{54} - x^9) \\
 \hline
 x^9 + 1
 \end{array}$$

$$\begin{array}{r}
 x^{45} + 1 \\
 \hline
 x^{54} - x^{27}
 \end{array}$$

$$\begin{array}{r}
 x^{45} + 1 \\
 \hline
 -(x^{45} + x^{36}) \\
 \hline
 -x^{36} + 1 \\
 \hline
 -(-x^{36} - x^{27}) \\
 \hline
 x^{27} + 1 \\
 \hline
 -(x^{27} + x^{18}) \\
 \hline
 -x^{18} + 1 \\
 \hline
 -(-x^{18} - x^9) \\
 \hline
 x^9 + 1
 \end{array}$$

$$\begin{array}{r}
 x^9 + 1 \\
 \hline
 x^{36} - x^{27} + x^{18} - x^9 + 1
 \end{array}$$

Le PGCD de deux polynômes est le dernier reste non nul dans l'algorithme d'euclide

Exercice n°7

$$\begin{array}{r}
 x^5 + 3x^4 + 7x^3 + 13x^2 + 12x + 4 \\
 - x^5 + x^4 + 6x^3 + 4x^2 + 8x \\
 \hline
 2x^4 + x^3 + 9x^2 + 4x + 4 \\
 - 2x^4 + 2x^3 + 12x^2 + 8x + 16 \\
 \hline
 -x^3 - 3x^2 - 4x - 12
 \end{array}$$

$$\begin{array}{r}
 x^4 + x^3 + 6x^2 + 4x + 8 \\
 - x+2
 \end{array}$$

$$\begin{array}{r}
 x^4 + x^3 + 6x^2 + 4x + 8 \\
 - x^4 + 3x^3 + 4x^2 + 12x \\
 \hline
 -2x^3 + 2x^2 - 8x + 8 \\
 2x^3 - 6x^2 - 8x - 24 \\
 \hline
 8x^2 + 32
 \end{array}$$

$$\begin{array}{r}
 -x^3 - 3x^2 - 4x - 12 \\
 - x+2
 \end{array}$$

$$\begin{array}{r}
 -x^3 - 3x^2 - 4x - 12 \\
 - x^3 - 4x \\
 \hline
 -3x^2 - 12 \\
 0
 \end{array}$$

$$\begin{array}{r}
 x^2 + 4
 \end{array}$$

$$\text{PGCD} = x^2 + 4$$

Racine A =

$$\begin{array}{r}
 x^4 + x^3 + 6x^2 + 4x + 8 \\
 - x^4 + 4x^2 \\
 \hline
 x^3 + 2x^2 + 4x + 8 \\
 - x^3 - 4x \\
 \hline
 2x^2 + 8 \\
 - 2x^2 + 8 \\
 \hline
 0
 \end{array}$$

$$\begin{array}{r}
 x^2 + 4
 \end{array}$$

$$* A(x^2 + 4)(x^2 + x + 2)$$

Racine de  $A \Rightarrow \lambda_1 = 2i$ ,  $\lambda_2 = -2i$  et  $\frac{-1-i\sqrt{7}}{2}$

$$\frac{-1+i\sqrt{7}}{2}$$

$$\begin{array}{r}
 x^5 + 3x^4 + 7x^3 + 13x^2 + 12x + 4 \\
 - x^5 \quad + 4x^3 \\
 \hline
 3x^4 + 3x^3 + 13x^2 + 12x + 4 \\
 - 3x^4 \quad + 12x^2 \\
 \hline
 3x^3 + x^2 + 12x + 4 \\
 - 3x^3 \quad + 12x \\
 \hline
 x^2 \quad + 4 \\
 - x^2 \quad + 4 \\
 \hline
 0
 \end{array}$$

$$\left| \begin{array}{l}
 x^2 + 4 \\
 x^3 + 3x^2 + 3x
 \end{array} \right.$$

$$B = (x^2 + 4)(x^3 + 3x^2 + 3x)$$

$$B = \{-2i, 2i, -1\}$$

$$A=0 \Leftrightarrow x^2+4=0 \text{ ou } x^2+x+2=0$$

$$\Delta = -16$$

$$\Delta = -7$$

$$x_1 = \frac{-i\sqrt{16}}{2} \quad x_2 = \frac{i\sqrt{16}}{2} \quad x_3 = \frac{-1-i\sqrt{7}}{2} \quad x_4 =$$

$$\frac{-1+i\sqrt{7}}{2}$$

$$B = (x^2 + 4)(x^3 + 3x^2 + 3x + 1)$$

$$\Leftrightarrow B = (x^2 + 4)(x + 1)^3$$

$$x_1 =$$

$$x_2 =$$

$$x_3 = -1$$

(racines triples)

$$B = A Q_1 + R_1 \Rightarrow R_1 = B - A Q_1$$

$$A = R_1 Q_2 + R_2 \Rightarrow R_2 = A - R_1 Q_2 = A - (B - A Q_1) Q_2$$

$$R_2 = -B Q_2 + A(1 + Q_1 Q_2)$$

$$A = (x^2 + 4)(x^2 + x + 2)$$

$$B = (x^2 + 4)(x + 1)^3$$

$$x^2 + 4 = A\left(-\frac{1}{8}x^2 + \frac{5}{8}\right) + B\left(\frac{x}{8} - \frac{2}{8}\right)$$

Exercice n°4

$$P_n = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!} + \dots + \frac{x(x+1)\dots(x+n)}{(n+1)!}$$

$$P_2 = 1 + \frac{x}{1!} + \frac{x(x+1)}{2!}$$

$$= 1 + x + \frac{x^2 + x}{2} = \frac{2 + 3x + x^2}{2}$$

$$P_1 \Rightarrow x_1 = -1 \\ x_2 = -2$$

$$P_1 = (1+x) \left[ 1 + \frac{x}{2!} \right] = \frac{(1+x)(2+x)}{2!}$$

$$P_2 = P_1 + \frac{x(x+1)(x+2)}{3!} = \frac{(1+x)(2+x)}{2!} + \frac{x(x+1)(x+2)}{3!}$$

$$= \frac{(1+x)(2+x)}{2!} \left[ 1 + \frac{x}{3} \right] = \frac{(1+x)(2+x)(3+x)}{3!}$$

$$P_n = \frac{(1+x)(2+x)\dots(m+1+x)}{(m+1)!}$$

① 1<sup>e</sup> rang  $n=1$  vérifié

② On suppose vrai jusqu'à l'ordre  $m$ .

③ On démontre à l'ordre  $m+1$

$$\begin{aligned}
 P_{n+1} &= P_n \cdot \frac{x(x+1) \dots (x+n+1)}{(n+2)!} \\
 &= \frac{(1+x)(2+x) \dots (n+1)+x}{(n+1)!} + \frac{x(x+1) \dots (x+n+1)}{(n+2)!} \\
 &= P_n \left( 1 + \frac{x}{n+2} \right) \\
 &= P_n \left( \frac{n+2+x}{n+2} \right)
 \end{aligned}$$

$x^{n+1} = 0$

### B. Polynômes à coefficients dans $\mathbb{Z}/2\mathbb{Z}$

$$\begin{aligned}
 & x^5 + x^4 + 1 + x^4 + x^3 + x + 1 \\
 &= x^4 + x^2 + x \\
 & (x^3 + x^2 + 1)(x^4 + x^3 + x + 1) \\
 &= x^5 + x^4 + x^3 + x^3 + x^2 + x + x^2 + x + 1 \\
 & \text{AUX } = x^5 - x^4 + \underbrace{(1+1)x^3}_{0} + \underbrace{(1+1)x^2}_{0} + \underbrace{(1+1)x}_{0} + 1 \\
 &= x^5 + x^4 + 1
 \end{aligned}$$

+	0	1
0	0	1
1	1	0

0 élément neutre  
symétrique de 0  
 $1=1$

- = + symétric

$$\begin{array}{r}
 x^5 + x^4 + 1 \\
 + (x^5 + x^3 + x) \\
 \hline
 x^4 + x^3 + x + 1 \\
 x^4 + x^2 + x + 1 \\
 \hline
 x^3 + x^2 + x
 \end{array}
 \quad \left| \begin{array}{l} x^4 + x^2 + 1 \\ x + 1 \end{array} \right.$$

$$\begin{array}{r} x^4 + x^2 + 1 \\ x^4 + x^3 + x^2 \\ \hline x^3 + 1 \\ x^3 + x^2 + x \\ \hline x^2 + x + 1 \end{array}$$

$$\begin{array}{r} x^3 + x^2 + x \\ x^3 + x^2 + 1 \\ \hline 0 \\ \boxed{\text{PGCD} = x^2 + x + 1} \end{array}$$

Racine d'un polynôme

### Exercice n°5

On pose  $x_0$ , une racine multiple d'ordre  $k$  de  $P(x)$ .

$$P(x) = (x - x_0)^k Q(x), \text{ avec } Q(x_0) \neq 0$$

$$P'(x) = k(x - x_0)^{k-1} Q(x) + (x - x_0)^k Q'(x)$$

$$\begin{matrix} & 0 & 0 \\ & 0 & 2 \\ 0 & 0 & 0 \\ & 2 & 0 \\ & 2 & 2 \\ & 2 & 0 \\ & 2 & 2 \\ & 2 & 0 \end{matrix}$$

$$P(x) = ax^{n+2} + bx^{n+1}$$

est divisible par  $(x-1)^2$  pour  $n \in \mathbb{N}$  si:

$$P(x) = (x-1)^2 \cdot Q(x)$$

$$P(1) = ax \cdot 1^{n+2} + bx \cdot 1^n + 1 = 0$$

$$\Leftrightarrow a+b+1 = 0$$

$$P'(x) = a(n+1)x^n + bn x^{n-1}$$

$$P'(1) \quad a(n+1) + bn = 0$$

$$\begin{cases} a+b+1=0 \\ a(n+1)+bn=0 \end{cases} \Leftrightarrow \begin{cases} a=-1-b \\ (-1-b)(n+1)+bn=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a=-1-b \\ -1n - 1 - bn - b + bn = 0 \end{cases} \Leftrightarrow \begin{cases} a=-1-b \\ -n - 1 - b = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a=-1-b \\ b=-1-n \end{cases} \Leftrightarrow \begin{cases} a=-1+1+n \\ b=-1-n \end{cases} \Leftrightarrow \begin{cases} a=n \\ b=-1-n \end{cases}$$

donc  $P(x)$  divisible par  $(x-1)^2$  pour  $a=n$   
et  $b=-1-n$ .