T. D. MDR

T.D. 1

Ex. 1

$$\begin{pmatrix} 1 & 2 & 3\lambda \\ 2 & 6 & 0 \\ 3\lambda & 0 & 3 \end{pmatrix}$$

$|1| = 1 > 0$

$\begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = 6 - 4 = 2 > 0$

1° méthode

$(y/x) = (x/y)$: symétrique

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

$(x/x) = x_1^2 + 6x_2^2 + 3x_3^2 + 4x_1x_2 + 6\lambda x_1 x_3$

$= (x_1 + 2x_2 + 3\lambda x_3)^2 + 2x_2^2 - 9\lambda^2 x_3^2$

$- 12\lambda x_2 x_3 + 3x_3^2$

$= (x_1 + 2x_2 + 3\lambda x_3)^2 + 2(x_2^2 - 6\lambda x_2 x_3) - 9\lambda^2 x_3^2 + 3x_3^2$

$= (x_1 + 2x_2 + 3\lambda x_3)^2 + 2(x_2 - 3\lambda x_3)^2 + (3 - 27\lambda^2) x_3^2$

$3 - 27\lambda^2 > 0 \qquad \frac{1}{9} - \lambda^2 > 0 \qquad \left(\frac{1}{3} - \lambda\right)\left(\frac{1}{3} + \lambda\right) > 0$

$\lambda \in \left]-\frac{1}{3} ; \frac{1}{3}\right[$

2° méthode

$A = \begin{pmatrix} 1 & 2 & 3\lambda \\ 2 & 6 & 0 \\ 3\lambda & 0 & 3 \end{pmatrix}$

$A_1 = 1 > 0$

$A_2 = \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} = 2 > 0$

$A_3 = \begin{vmatrix} 1 & 2 & 3\lambda \\ 2 & 6 & 0 \\ 3\lambda & 0 & 3 \end{vmatrix} = 6(1 - 9\lambda^2) > 0$

$\iff \lambda \in \left]-\frac{1}{3} ; \frac{1}{3}\right[$

$6 - 54\lambda^2$

$\Delta = -4 \times 6 \times -54$

$= 24 \times 54$

$6\sqrt{54} < x < 13$

Ex. 2

$(x|y) = \left(\frac{1}{\sqrt{3}} x_1 - x_2 + x_3\right) \times \left(\frac{1}{\sqrt{3}} x_1 - x_2 + x_3\right) + (x_2 - x_3) \times (y_2 - y_3) + 3 x_3 y_3$

$(x|x) = \left(\frac{1}{\sqrt{3}} x_1 - x_2 + x_3\right)^2 + (x_2 - x_3)^2 + 3x_3^2$: symétrique

$(x/x) = 0 \Rightarrow x = 0 \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$: définie

$\begin{cases} \frac{1}{\sqrt{3}} x_1 - x_2 + x_3 = 0 \\ x_2 - x_3 = 0 \\ x_3 = 0 \end{cases} \qquad \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$

1.3 a, b réels $a < b$ f fonct. positive sur $[a, b]$ $f \not\equiv 0$

$$\varphi : \mathbb{R}[X] \times \mathbb{R}[X] \longrightarrow \mathbb{R}$$
$$(P, Q) \longmapsto \int_a^b f(t) P(t) Q(t) \, dt$$

$\int f + g = \int f + \int g$

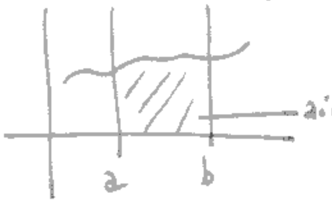
$\int \alpha f = \alpha \int f$

$$\varphi(P, P) = \int_a^b f(t) P^2(t) \, dt$$

$\int_a^b f(t) (\alpha P_1 + \beta P_2)(x) \, \varphi(x) \, dx$
$= \alpha \int_a^b f P_1(x) Q_1(x) \, dx + \beta \int_a^b f P_2(x) Q_2(x) \, dx$

bilinéarité de l'intégrale –

L'intégrale d'une fonction positive est positive.



$\varphi(P, P) = 0 \overset{?}{\Longrightarrow} P = 0$

$\varphi(P, P) = 0 \Longrightarrow \int_a^b f(t) P^2(t) \, dt = 0 \Longrightarrow f(t) P^2(t) = 0$ sur $[a, b]$

$\exists \, [c, d] \subset [a, b] : P(t) = 0$ donc P admet une infinité de racines,

donc $P \equiv 0$

1.4 $\mathbb{R}_2[X]$: ensemble des polynomes de degré inférieur ou égal à 2

$d° \leq 2$

$$\langle P, Q \rangle = \int_{-1}^{1} P(x) Q(x) \, dx \qquad \text{Base de } \mathbb{R}_2[X] \text{ est } (1, X, X^2) \text{ (can)}$$

sym. bili.

$\langle 1, 1 \rangle = \int_{-1}^{1} dX = 2$

$\langle 1, X \rangle = \langle X, 1 \rangle = \int_{-1}^{1} X \, dX = 0$

$\langle X, X \rangle = \langle 1, X^2 \rangle = \langle X^2, 1 \rangle = \int_{-1}^{1} X^2 \, dX = \frac{2}{3}$

$\langle X, X^2 \rangle = \langle X^2, X \rangle = \int_{-1}^{1} X^3 \, dX = 0$

$\langle X^2, X^2 \rangle = \int_{-1}^{1} X^4 \, dX = \frac{2}{5}$

$\left[\frac{x^5}{5} \right]_{-1}^{1} = \frac{1}{5} + \frac{1}{5}$

$$A = \begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix} \qquad A_1 = 2 > 0 \qquad A_2 = \frac{4}{3} > 0 \qquad A_3 = \frac{8}{15} - \frac{8}{27} > 0$$

1.5

$\langle 1, 1 \rangle = \int_0^{\pi} \sin(t) \, dt = \left[-\cos(t) \right]_0^{\pi} = -(-1) - (-1) = 2$

$\langle 1, X \rangle = \langle X, 1 \rangle = \int_0^{\pi} t \sin(t) \, dt$

$u(t) = t \quad u'(t) = 1$
$v'(t) = \sin(t) \quad v(t) = -\cos(t)$

$\Longrightarrow \int_0^{\pi} t \sin(t) \, dt = -\left[t \cos t \right]_0^{\pi} + \int_0^{\pi} \cos t \, dt$

$= -\pi \cos \pi + \left[\sin t \right]_0^{\pi} = \pi + (\sin \pi - \sin 0) = \pi$

T. D. HDR

1.5 (suite)

$< t,t > = <1, t^2> = <t^2, 1> = \int_0^{\pi} t^2 \sin(t)\,dt$ $\quad u(t) = t^2 \quad u'(t) = 2t$ $\quad v'(t) = \sin(t) \quad v(t) = -\cos(t)$

$$\int_0^{\pi} t^2 \sin(t)\,dt = \left[-t^2\cos(t)\right]_0^{\pi} + \int_0^{\pi} 2t \cos t\,dt$$

$$= -\left(\pi^2 \cos \pi\right) + 2\int t\cos t\,dt$$

$$= \pi^2 + 2\underbrace{\int_0^{\pi} t\cos t\,dt}_{A}$$

A) $u(t) = t \quad u'(t) = 1$

$v'(t) = \cos t \quad v(t) = \sin t$

$$A = \left[t\sin t\right]_0^{\pi} - \int_0^{\pi} \sin t\,dt$$

$$= \pi \sin \pi + \left[\cos t\right]_0^{\pi}$$

$$= (-1-1) = -2$$

$$\boxed{\int_0^{\pi} t^2 \sin t = \pi^2 - 4}$$

$B = < X^2, X > = \int_0^{\pi} t^3 \sin t\,dt$ $\quad u(t) = t^3 \quad u'(t) = 3t^2$ $\quad v'(t) = \sin t \quad v(t) = -\cos t$

$$B = -\left[t^3\cos t\right]_0^{\pi} + \int_0^{\pi} 3t^2\cos t\,dt$$

$$= -\left[t^3\cos t\right]_0^{\pi} + 3\int_0^{\pi} t^2\cos t\,dt$$

$u(t) = t^2 \quad u'(t) = 2t$

$v'(t) = \cos t \quad v(t) = \sin t$

$$B = \pi^3 + 3\left[t^2\sin t\right]_0^{\pi} - 3\int_0^{\pi} 2t\sin t\,dt$$

$$= \pi^3 - 6\int_0^{\pi} t\sin t\,dt$$

$$= \boxed{\pi^3 - 6\pi}$$

$$C = \int_0^{\pi} t^4 \sin t\,dt = \left[-t^4\cos t\right]_0^{\pi} + 4\int_0^{\pi} t^3\cos t\,dt$$

$$\int_0^{\pi} t^3\cos t\,dt = \left[t^3\sin t\right]_0^{\pi} - 3\int_0^{\pi} t^2\sin t\,dt$$

$$C = \pi^4 - 12(\pi^2 - 4) = \boxed{\pi^4 - 12\pi^2 + 48}$$

$$A = \begin{pmatrix} 2 & \pi & \pi^2-4 \\ \pi & \pi^2-4 & \pi^3-6\pi \\ \pi^2-4 & \pi^3-6\pi & \pi^4-12\pi^2+48 \end{pmatrix}$$

$A_1 = 2 \quad > 0$

$A_2 = \pi^2 - 8 \quad > 0$

$A_3 = 2(\pi^2-4)(\pi^4-12\pi^2+48) + \pi(\pi^3-6\pi)(\pi^2-4)$
$+ \pi(\pi^2-4)(\pi^3-6\pi) - \left[\pi^2 \times (\pi^4 - 12\pi^2+48) + 2(\pi^3-6\pi)^2 + (\pi^2-4)^3\right]$

$A_3 = \pi^6 - 12\pi^4 + 48\pi^2 - 4\pi^4 + 48\pi^2 - 192 \dots$

$A_3 = 72\pi^2 - 4\pi^4 - 320 \simeq 0,97 > 0$

1.6 à la maison

1.7 $(1-x)^2 + (x-y)^2 + (y-z^2) + z^2 = \frac{1}{4}$

Dans $\mathbb{R}^4$: $(1,1,1,1)$ et $(1-x, x-y, y-z, z)$

$$\left((1,1,1,1)/(1-x, x-y, y-z, z)\right) = \left((1-x) + (x-y) + (y-z) + z\right)^2$$

$$\leq 4\left((1-x)^2 + (x-y)^2 + (y-z)^2 + z^2\right) \leftarrow \text{inégalité de Cauchy-Schwarz}$$

$$\frac{1}{4} \leq (1-x)^2 + (x-y)^2 + (y-z)^2 + z^2$$

→ égalité dans le cas où la famille est liée.

$$\begin{cases} 1-x = k \\ x-y = k \\ y-z = k \\ z = k \end{cases} \qquad 1-x = x-y = y-z = z \qquad \begin{cases} x = \frac{3}{4} \\ y = \frac{1}{2} \\ z = \frac{1}{4} \end{cases}$$

2.1

$N_1 : \mathbb{R}^2 \to \mathbb{R}$

$(x_1, x_2) \mapsto |x_1| + |x_2|$

– $\forall (x_1, x_2) \quad N_1(x_1, x_2) = |x_1| + |x_2| \geq 0$

– $N_1(x_1, x_2) = 0 \overset{?}{\Rightarrow} (x_1, x_2) = (0,0)$

$N_1(x_1, x_2) = 0 \Rightarrow |x_1| + |x_2| = 0 \Rightarrow |x_1| = |x_2| = 0$
$\Rightarrow x_1 = x_2 = 0$

$\Rightarrow (x_1, x_2) = (0,0)$

– $\forall \lambda \in \mathbb{R} \quad N_1(\lambda(x_1, x_2)) = N_1(\lambda x_1, \lambda x_2) = |\lambda x_1| + |\lambda x_2|$
$= |\lambda||x_1| + |\lambda||x_2|$
$= |\lambda|(|x_1| + |x_2|) = |\lambda| N_1(x_1, x_2)$

$\forall (x_1, x_2)(y_1, y_2) \in \mathbb{R}^2$

$N_1((x_1, x_2) + (y_1, y_2)) = N_1(x_1 + y_1, x_2 + y_2) = |x_1 + y_1| + |x_2 + y_2|$
$\leq |x_1| + |y_1| + |x_2| + |y_2|$
$= |x_1| + |x_2| + |y_1| + |y_2| = N_1(x_1, x_2) + N_1(y_1, y_2)$

2.2

2.2.1

$N : \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \mapsto \sup \dfrac{|xt+y|}{1+t^2}$

* $\forall t \in \mathbb{R} \quad 1+t^2 \neq 0 \quad \dfrac{xt+y}{1+t^2} \underset{+\infty}{\sim} \dfrac{x}{t} \xrightarrow[t \to +\infty]{} 0$

donc $\forall (x,y) \in \mathbb{R}^2$

$\forall t \in \mathbb{R} \quad \dfrac{|xt+y|}{1+t^2} \geq 0$ donc $N(x,y) \in \mathbb{R}^+$

* $\forall (x,y) \in \mathbb{R}^2 : N(x,y) = 0 \Rightarrow (x,y) = (0,0)$

$N(x,y) = 0 \Rightarrow \sup\limits_{t \in \mathbb{R}} \dfrac{|xt+y|}{1+t^2} = 0 \Rightarrow \forall t \in \mathbb{R} \ \dfrac{|xt+y|}{1+t^2} = 0$

$\Rightarrow \forall t \in \mathbb{R} \ |xt+y| = 0 \Rightarrow \forall t \in \mathbb{R} \ xt+y = 0$

ex. : $\Rightarrow \begin{cases} t=0 & y=0 \\ t=1 & x+y=0 \end{cases} \Rightarrow (x,y) = (0,0)$

* $\forall \lambda \in \mathbb{R} \ \forall (x,y) \in \mathbb{R}^2 \ N(\lambda(x,y)) = N(\lambda x, \lambda y) = \sup\limits_{t \in \mathbb{R}} \dfrac{|t\lambda x + \lambda y|}{1+t^2}$

$= \sup\limits_{t \in \mathbb{R}} \dfrac{|\lambda||tx+y|}{1+t^2} = |\lambda| \sup\limits_{t \in \mathbb{R}} \dfrac{|tx+y|}{1+t^2} = |\lambda| N(x,y)$

$* \ \forall (x,y) \ (x',y') \in \mathbb{R}^2 \quad N((x,y)+(x',y')) = N(x+x', y+y') = \sup_{t\in\mathbb{R}} \frac{|t(x+x')+(y+y')|}{1+t^2}$

$$= \sup_{t\in\mathbb{R}} \frac{|tx+y+tx'+y'|}{1+t^2} \leq \sup_{t\in\mathbb{R}} \frac{|tx+y|+|tx'+y'|}{1+t^2}$$

$$\leq \sup_{t\in\mathbb{R}} \frac{|tx+y|}{1+t^2} + \sup_{t\in\mathbb{R}} \frac{|tx'+y'|}{1+t^2}$$

$$= N(x,y) + N(x',y')$$

2.2.2. $\overline{B}((0,0),1) = \left\{(x,y) \in \mathbb{R}^2 : N(x,y) \leq 1\right\}$

$N(x,y) \leq 1 \iff \sup_{t\in\mathbb{R}} \frac{|xt+y|}{1+t^2} \leq 1$

$\iff \forall t \in \mathbb{R} \quad \frac{|xt+y|}{1+t^2} \leq 1$

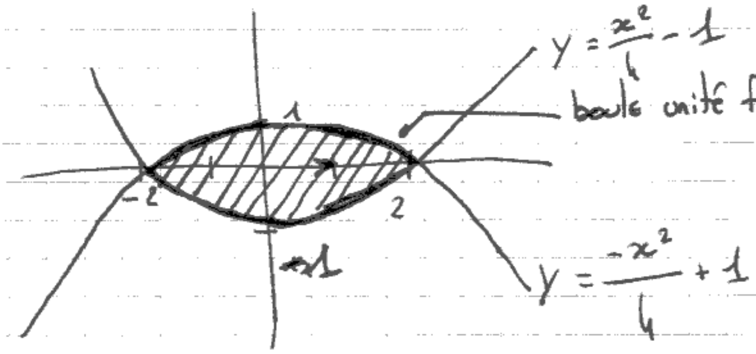
$\iff \forall t \in \mathbb{R} \quad |xt+y| \leq 1+t^2$

$\iff \forall t \in \mathbb{R} \quad -1-t^2 \leq xt+y \leq 1+t^2$

$\iff \forall t \in \mathbb{R} \quad 0 \leq t^2+xt+y+1 \text{ et } 0 \leq t^2-xt+1-y$

$\iff x^2-4(y+1) \leq 0 \text{ et } (-x)^2-4(1-y) \leq 0$

$\iff \frac{x^2}{4}-1 \leq y \text{ et } y \leq \frac{-x^2}{4}+1$



2.3. ~~...~~ Idem que le 2

~~2~~2.4. 2.4.1. $d : E^2 \to \mathbb{R}$

$$\begin{cases} (x,y) \mapsto 0 & \text{si } x = y \\ (x,y) \mapsto 1 & \text{si } x \neq y \end{cases}$$

1. ~~Démontrer que d est~~

a) On a $x = y \Rightarrow d(x,y) = 0$ (1)
si $x \neq y \Rightarrow d(x,y) = 1$

On a $d(x,y) = 0$ si $x \neq y$ on a $d(x,y) = 1$
dans ce cas $1 = 0$ (absurde) donc $x = y$
$\Rightarrow d(x,y) = 0 \Rightarrow x = y$ (2)

d'après (1) et (2)

$$d(x,y) = 0 \iff x = y$$

b) $d(x,y)=0 \iff x=y$

$d(y,x)=0 \iff \cancel{y}=x$

même modèle pour $d(y,x)=1$

donc $d(x,y)=d(y,x) \quad \forall (x,y) \in E^2$

c) Cas 1 : $x=y=z$

$$\left.\begin{array}{l} d(x,y)=0 \\ d(y,z)=0 \\ d(x,z)=0 \end{array}\right\} \qquad d(x,z) \leq d(x,y)+d(y,z)$$

Cas 2 : $x \neq y$ et $y=z$

$$\left.\begin{array}{l} d(x,y)=1 \\ d(x,z)=1 \\ d(y,z)=0 \end{array}\right\} d(x,z) \leq d(x,y)+d(y,z)$$

Cas 3 : $x=z$ et $y \neq z$

$$\left.\begin{array}{l} d(x,z)=0 \\ d(x,y)=1 \\ d(y,z)=1 \end{array}\right\} d(x,z) \leq d(x,y)+d(y,z)$$

Cas 4 : $x=y$ et $z \neq y$

$$\left.\begin{array}{l} d(x,y)=0 \\ d(y,z)=1 \\ d(x,z)=1 \end{array}\right\}$$

Cas 5 : $x \neq z$ et $y=z$

$$\left.\begin{array}{l} d(x,y)=1 \\ d(y,z)=0 \\ d(x,z)=1 \end{array}\right\}$$

Cas 6 : $x \neq y \neq z$

$$\left.\begin{array}{l} d(x,y)=1 \\ d(x,z)=1 \\ d(y,z)=1 \end{array}\right\}$$

On a dans les 6 cas $d(x,z) \leq d(x,y)+d(y,z)$

et a), b), c) démontrés donc d est une distance discrète sur E.

2.4.2. Boule unité de centre a et de rayon 1

$= \{x \in E \mid d(a,x) < 1\}$

$= \{a\}$

~~1.3~~ 1.6 $1, X, X^2$ — Avec Matrices

$(P/Q) = P(0)Q(0) + P'(0)Q'(0) + P''(0)Q''(0)$

f	f'	f''
1	0	0
X	1	0
X^2	$2X$	2

$$\begin{array}{c|ccc} & 1 & X & X^2 \\ \hline 1 & 1 & 0 & 0 \\ X & \boxed{0} & 1 & 0 \\ X^2 & 0 & 0 & \boxed{4} \end{array}$$

$P(X) = X \quad Q(X) = 1$

$A_1 = 1 > 0$
$A_2 = \cancel{1}\ 1 > 0$
$A_3 = 4 > 0$

$P(X) = X^2 \quad Q(X) = X^2$

Sans mat. $(P,P) = P^2(0) + P'^2(0) + P''^2(0)$

$= c^2 + b^2 + 4a^2 \geq 0$

$P(X) = aX^2 + bX + c$
$P'(X) = 2aX + b$
$P''(X) = 2a$

$(P,P) = 0 \Rightarrow c^2 = b^2 = 4a^2 = 0$
$\Rightarrow a = b = c = 0 \Rightarrow P = 0$

T.D. 3

~~3.1~~ 3.2 1. $f(x,y) = \dfrac{x^5 y^3}{x^6 + y^4}$

$D_f = \{(x,y) \in \mathbb{R}^2 : x^6 + y^4 \neq 0\} = \mathbb{R}^2 \setminus \{(0,0)\}$

$\forall r > 0$
$\forall (x,y) \in B(0,r) \qquad \|(x,y)\| < r$

$\|(x,y)\| = \sup(|x|, |y|) \rightarrow$ Fonctionne avec toutes les normes mais on prend la plus simple

si $|x| \leq |y|$ $\quad |f(x,y)| = \left|\dfrac{x^5 y^3}{x^6 + y^4}\right| \leq \dfrac{|x^5||y^3|}{x^6 + y^4} \leq \dfrac{|y^5||y^3|}{y^4} = |y^4| \leq r^4 \underset{r \to 0}{\longrightarrow} 0$

il faut majorer le num. et minorer le dénom.

$0 \leq x^6$ donc $y^4 \leq x^6 + y^4$ donc $\dfrac{1}{x^6 + y^4} \leq \dfrac{1}{y^4}$

Si $|y| \leq |x|$ $\quad |f(x,y)| = \dfrac{|x^5 y^3|}{x^6 + y^4} \leq \dfrac{|x^8|}{x^6} = |x^2| < r^2 \underset{r \to 0}{\longrightarrow} 0$

T. D. 3 (suite)

3.1.2.

$$f(x,y) = \frac{((x-1)^2+y^2)\ln((x-1)^2+y^2)}{|x|+|y|}$$

$$D_f = \left\{(x,y) \in \mathbb{R}^2 : (x-1)^2+y^2 > 0 \text{ et } |x|+|y| \neq 0\right\}$$

$(x-1)^2+y^2 > 0 \iff x \neq 1$ et $y \neq 0$; $(x,y) \neq (1,0)$

$(x,y) \neq (0,0)$

$$\boxed{\ln x^p = p \ln x}$$

$$D_f = \mathbb{R}^2 - \{(1,0),(0,0)\}$$

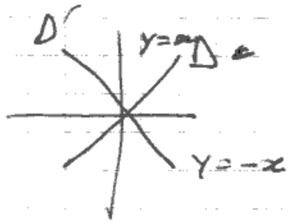
En $(0,0)$: $f(x,0) = \dfrac{(x-1)^2\ln((x-1)^2)}{|x|} = \dfrac{(x-1)^2\ln(x-1)^2}{|x|} = \dfrac{2(x-1)^2\ln|1-x|}{|x|}$

$\underset{0}{\sim} 2\dfrac{x}{|x|}$ ← DL

$f(x,0) < \begin{matrix} +2 & \text{si } x \text{ positif} \\ -2 & \text{si } x \text{ négatif} \end{matrix}$ donc pas de limite en $(0,0)$.

→ En $(1,0)$?

3.1.3.



$$f(x,y) = \frac{1+x+y}{x^2-y^2}$$

$D_{ef} = \{(x,y) \in \mathbb{R}^2 / x^2-y^2 \neq 0\}$

$x^2-y^2=0 \iff (x-y)(x+y)=0 \iff y=x$ ou $y=-x$

$= \mathbb{R}^2 \setminus \{\Delta \cup \Delta'\}$ avec $\Delta : y=x$ et $\Delta' : y=-x$

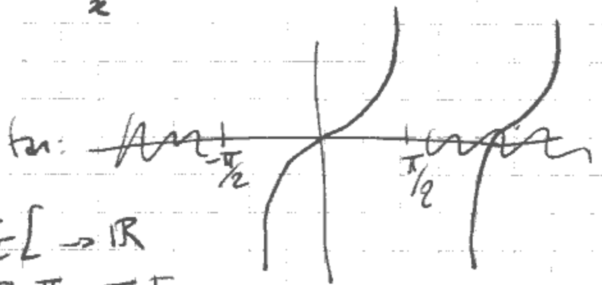
En $(0,0)$ $f(x,0) = \dfrac{1+x}{x^2} \xrightarrow[x\to 0]{} +\infty$ ← x négligeable devant 1

Pas de continuité donc pas de limite finie en $(0,0)$.

et $f(0,y) = \dfrac{1+y}{-y^2} \xrightarrow[y\to 0]{} -\infty$

Donc pas de limite du tout.

3.1.4. $f(x,y) = \arctan\dfrac{x}{y} + \arctan\dfrac{y}{x}$ $\qquad \tan : \mathbb{R}\setminus\left\{\frac{\pi}{2}+k\pi\right\} \to \mathbb{R}$



$\tan : \left]\frac{-\pi}{2} ; \frac{\pi}{2}\right[\to \mathbb{R}$

$\arctan : \mathbb{R} \to \left]\frac{-\pi}{2} ; \frac{\pi}{2}\right[$

$D_{ef} = \{(x,y) \in \mathbb{R}^2 : y \neq 0 \text{ et } x \neq 0\}$

$\quad = \mathbb{R}^{*2}$

"$\arctan u + \arctan \frac{1}{u} = \underline{\text{signe}}(u) \times \frac{\pi}{2} \quad \forall u \in \mathbb{R}^*$"

$\varphi :]-\infty ; 0[\cup]0 ; +\infty[\longrightarrow \mathbb{R}$

$\quad u \longmapsto \arctan u + \arctan \frac{1}{u}$

$$\varphi'(u) = \frac{1}{1+u^2} + \frac{\left(\frac{1}{u}\right)'}{1+\left(\frac{1}{u}\right)^2}$$

$$= \frac{1}{1+u^2} + \frac{\frac{-1}{u^2}}{\frac{u^2+1}{u^2}}$$

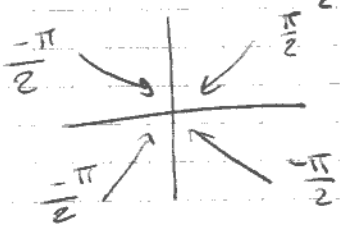
$$= \frac{1}{1+u^2} - \frac{1}{1+u^2} = 0$$

Si $u > 0$ $\quad u = 1$ (car image de n'importe quelle valeur positive)

$\varphi(1) = \arctan 1 + \arctan 1 = \frac{2\pi}{4} = \frac{\pi}{2}$

si $u < 0$ $\quad u = -1$ $\quad \varphi(-1) = \arctan(-1) + \arctan(-1) = \frac{-\pi}{4} - \frac{\pi}{4} = \frac{-\pi}{2}$

$f(x,y) = \underline{\text{signe}}\left(\frac{x}{y}\right)\frac{\pi}{2}$ $\to +\frac{\pi}{2}$, $-\frac{\pi}{2}$



pas de limite en $(0,0)$

Ex. 2

$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \qquad D_f = \{(x,y) \in \mathbb{R}^2 / x^2 + y^2 \neq 0\} = \mathbb{R}^2 \setminus \{(0,0)\}$

$\left.\begin{array}{l} f(x,0) = 1 \\ f(0,y) = -1 \end{array}\right\} \neq$ donc pas de limite en $(0,0)$

$g(x,y) = x \arctan\left(\frac{y}{x}\right) \quad D_{ef} = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0\} = \mathbb{R}^* \times \mathbb{R}$

$|g(x,y)| \leq |x| \left|\arctan \frac{y}{x}\right| \leq \frac{\pi}{2}|x| \leq r \xrightarrow[r \to 0]{} 0$

Ex. 3

$\forall t \in \mathbb{R} \quad |\sin t| \leq |t|$

$\Longleftrightarrow \forall t \in \mathbb{R}^+ \quad -t \leq \sin t \leq t$

$\forall t \in [0 ; +\infty[\quad 0 \leq t - \sin t = \varphi(t)$ et $\forall t \in [0 ; +\infty[\quad 0 \leq t + \sin t = \psi(t)$

$\varphi'(t) = 1 - \cos t \geqslant 0$

t	0		$+\infty$
φ'		$+$	
φ	0	$\nearrow$	

$\psi'(t) = 1 + \cos t \geqslant 0$

t	0		$+\infty$
ψ'		$+$	
ψ	0	$\nearrow$	

$\forall\, t \in [0;+\infty[\quad \varphi(t) \geqslant 0 \qquad \forall\, t \in [0;+\infty[\quad \psi(t) \geqslant 0$

$\forall\, t \in \mathbb{R}$

$\operatorname{ch} t \geqslant 1 + \frac{t^2}{2} \iff \forall\, t \in \mathbb{R}\quad \underbrace{\operatorname{ch}(t) - 1 - \frac{t^2}{2}}_{= \varphi(t)} \geqslant 0$

$\operatorname{ch} t = \frac{e^t + e^{-t}}{2}$

$\varphi'(t) = \operatorname{sh}(t) - t \qquad \varphi''(t) = \operatorname{ch}(t) - 1 \geqslant 0$

car $\frac{e^t + e^{-t}}{2} \geqslant 1$ (car e^t croissante et $e^0 = 1$)

	t	$-\infty$		0		$+\infty$
sig	φ''			$+$		
var.	φ'		$\nearrow$	0	$\nearrow$	
sig	φ'		$-$	0	$+$	
var.	φ		$\searrow$	0	$\nearrow$	

$\varphi'(0) = 0$

$\varphi(0) = \operatorname{ch}(0) - 1 + \frac{0^2}{2}$
$= 0$

$\forall\, t \in \mathbb{R}\quad 1 - \frac{t^2}{2} \leq \cos t \leq 1 - \frac{t^2}{2} + \frac{t^4}{24}$

$\forall\, t \in \mathbb{R}\quad 0 \overset{?}{\leq} \varphi(t) = \cos t - 1 + \frac{t^2}{2} \qquad 0 \overset{?}{\leq} \psi(t) = \frac{t^4}{24} - \frac{t^2}{2} + 1 - \cos t$

$\varphi'(t) = -\sin t + t$

$\varphi''(t) = -\cos t + 1 \geqslant 0$

	t	$-\infty$		0		$+\infty$
sig	φ''			$+$		
var.	φ'		$\nearrow$	0	$\nearrow$	
sig	φ'		$-$	0	$+$	
	φ		$\searrow$	0	$\nearrow$	

$\varphi(0) = 0$

$\psi'(t) = \frac{t^3}{6} - t + \sin t$ $\quad\quad$ $\psi''(t) = \frac{t^2}{2} - 1 + \cos t \geqslant 0$
(démontré précédemment)

	t	$-\infty$	0	$+\infty$
s.	ψ''		$+$	
v.	ψ'	↗	0	↗
s.	ψ'	$-$	0	$+$
v.	ψ	↘	0	↗

$\psi(0) = 0$