Exercice 3

$\mathbb{R}^3$

$x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$

$\Rightarrow (x/y) = \left(\frac{1}{\sqrt{3}} x_1 - x_2 + x_3\right)\left(\frac{1}{\sqrt{3}} y_1 - y_2 + y_3\right) + (x_2 - x_3)(y_2 - y_3) + 3x_3y_3$

→ symétrique: on remplace x par y donc ici ça ne changeait rien donc c'est symétrique.

Si $(x/x) = 0 \Rightarrow x = 0$

or ici: $(x/x) = \left(\frac{1}{\sqrt{3}} x_1 - x_2 + x_3\right)^2 + (x_2 - x_3)^2 + 3x_3^2 = 0$

$$\Leftrightarrow \begin{cases} \frac{1}{\sqrt{3}} x_1 + x_2 + x_3 = 0 \\ x_2 - x_3 = 0 \\ x_3 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases} \Leftrightarrow x = 0$$

$\Rightarrow$ elle est donc définie.

Exercice 5

$\mathbb{R}_2[X]$ → ensemble des poly du 2nd degrés

$\langle P, Q \rangle = \int_{-1}^{1} P(X)\, Q(X)\, dX$

1) Montrons qu'elle est symétrique.

$\langle P, Q \rangle = \int_{-1}^{1} P(X) Q(X) dX = \int_{-1}^{1} Q(X) P(X) dX = \langle Q / P \rangle$

→ symétrique

2) Montrons la linéarité

$\forall (P_1, P_2, Q) \in E^3 \quad \forall (\alpha, \beta) \in \mathbb{R}^2$

$\langle \alpha P_1 + \beta P_2 | Q \rangle = \int_{-1}^{1} (\alpha P_1 + \beta P_2)(X)\, Q(X)\, dX$

$= \alpha \int_{-1}^{1} P_1(X) P(X) dX + \beta \int_{-1}^{1} P_2(X) Q(X) dX$

$= \alpha \langle P_1 / Q \rangle + \beta \langle P_2 | Q \rangle$

→ elle est donc bilinéaire

3) Base canonique de $\mathbb{R}_2[X]$ est $(1, X, X^2)$

$$M = \begin{pmatrix} \langle 1/1 \rangle & \langle 1/X \rangle & \langle 1/X^2 \rangle \\ \langle X/1 \rangle & \langle X/X \rangle & \langle X/X^2 \rangle \\ \langle X^2/1 \rangle & \langle X^2/X \rangle & \langle X^2, X^2 \rangle \end{pmatrix}$$

- $\langle 1/1 \rangle = \int_{-1}^{1} 1 \times 1 \, dx = [x]_{-1}^{1} = 2$

$$M = \begin{pmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{pmatrix}$$

- $\langle X/1 \rangle = \langle 1 \times X \rangle = \int_{-1}^{1} X \, dX = \left[\frac{X^2}{2}\right]_{-1}^{1} = 0$
- $\langle 1 | X^2 \rangle = \langle X^2/1 \rangle = \langle X/X \rangle = \int_{-1}^{1} X^2 \, dX = \left[\frac{1}{3}\right]_{-1}^{1} = \frac{2}{3}$
- $\langle X/X^2 \rangle = \langle X^2/X \rangle = \int_{-1}^{1} X^3 \, dX = \left[\frac{X^4}{4}\right]_{-1}^{1} = 0$
- $\langle X^2/X^2 \rangle = \int_{-1}^{1} X^4 \, dX = \left[\frac{X^5}{5}\right]_{-1}^{1} = \frac{2}{5}$.

$\Rightarrow \Pi_1 = 2 > 0$

$\Pi_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2/3 \end{vmatrix} = \frac{4}{3} > 0$

$\Pi_3 = 2 \begin{vmatrix} 2/5 & 0 \\ 0 & 2/5 \end{vmatrix} + \frac{2}{3} \begin{vmatrix} 0 & 2/3 \\ 2/3 & 0 \end{vmatrix} = \frac{8}{15}$

Exercice 6

$\mathbb{R}_2[X]$

$$\langle P, Q \rangle = \int_{-1}^{1} P(X) Q(X) \, dX$$

• Symétrie : $\langle P, Q\rangle = \int_0^\pi \sin x \, P(x) Q(x) \, dx$

$= \int_0^\pi \sin x \, Q(x) P(x) \, dx$

$= \langle Q / P \rangle$

=> symétrique

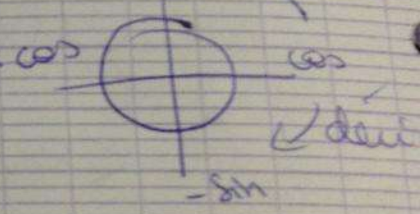
• bilinéarité : $\forall (P_1, P_2, Q) \in E^3 \ \forall (\alpha, \beta) \in \mathbb{R}^2$.

$\langle \alpha P_1 + \beta P_2 / Q \rangle = \int_0^\pi (\alpha P_1 + \beta P_2)(x) Q(x) \sin x \, dx$

$= \alpha \int_0^\pi P_1(x) P(x) \sin x \, dx + \beta \int_0^\pi P_2(x) Q(x) \sin x \, dx$

$= \alpha \langle P_1 / Q \rangle + \beta \langle P_2 / Q \rangle$

Base canonique de $\mathbb{R}_2[X]$ est



$$M = \begin{pmatrix} \langle 1/1\rangle & \langle 1/X\rangle & \langle 1/X^2\rangle \\ \langle X/1\rangle & \langle X/X\rangle & \langle X/X^2\rangle \\ \langle X^2/1\rangle & \langle X^2/X\rangle & \langle X^2/X^2\rangle \end{pmatrix}$$

• $\langle 1/1 \rangle = \int_0^\pi \sin X \, dX = [-\cos x]_0^\pi = 2$

• $\langle X/1 \rangle = \langle 1/X \rangle = \int_0^\pi X \sin X \, dX = \pi$

$\int (u.v)' = \int u'v + \int uv'$

$uv = \int u'v + \int uv'$

$\int u'v = uv - \int uv'$

→ IPP : $\int_0^\pi \underset{1}{x} \overset{-\cos x}{\sin x} \, dx = [-x \cos x]_0^\pi + \int_0^\pi \cos x \, dx$

$= \pi + [\sin x]_0^\pi = \pi$

• $\langle 1/X^2 \rangle = \langle X^2/1 \rangle = \langle X/X \rangle = \int_0^\pi X^2 \sin X \, dX$

$=$

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$$\int_0^\pi x^2 \sin x\,dx = [-x^2\cos x]_0^\pi + 2\int_0^\pi x\cos x\,dx$$
$$= \pi^2 + [2x\sin x]_0^\pi - 2\int \sin x\,dx$$
$$= \pi^2 - 2[-\cos x]_0^\pi$$
$$= \pi^2 - 2\times(1+1)$$
$$= \pi^2 - 4.$$

- $\langle X/X^2\rangle = \langle X^2/X\rangle = \int_0^\pi x^3 \sin x\,dx$

$$= [-X^3\cos X]_0^\pi + \int_0^\pi 3X^2\cos X\,dX$$
$$= \pi^3 + [3X^2\sin X]_0^\pi - \int_0^\pi 6X\sin X\,dX$$
$$= \pi^3 + [-6X\cos X]_0^\pi + 6\int_0^\pi \cos X\,dX$$
$$= \pi^3 - 6\pi + 6[\sin X]_0^\pi$$
$$= \pi^3 - 6\pi$$

- $\langle X^2/X^2\rangle = \int_0^\pi X^4 \sin X\,dX$

$$\int_0^\pi X^4 \sin X\,dX = [-X^4\cos X]_0^\pi + \int_0^\pi 4X^3\cos X\,dX$$
$$= \pi^4 + [4X^3\sin X]_0^\pi - \int_0^\pi 12X^2\sin X\,dX$$
$$= \pi^4 - [-12X\cos X]_0^\pi + \int_0^\pi 24X\cos X\,dX$$
$$= \pi^4 - 12\pi^2 + [-24\cos X]_0^\pi$$
$$= \pi^4 - 12\pi^2 + 48.$$

$$M = \begin{pmatrix} 2 & \pi & \pi^2-4 \\ \pi & \pi^2-4 & \pi^3-6\pi \\ \pi^2-4 & \pi^3-6\pi & \pi^4-12\pi^2+48 \end{pmatrix}$$

$M_1 = 2 > 0$ $\quad M_2 = \begin{vmatrix} 2 & \pi \\ \pi & \pi^2-4 \end{vmatrix} = 2\times(\pi^2-4) - \pi\times\pi$
$$= 2\pi^2 - 8 - \pi^2$$
$$= \pi^2 - 8 > 0$$

$\pi_3 = 72\pi^2 - 4\pi^4 - 320 > 0$

Exercice 9

Résoudre $(1-x)^2 + (x-y)^2 + (y-z)^2 + z^2 = \frac{1}{4}$.

$(x, y, z) \in \mathbb{R}^3$. $(\vec{u}\cdot\vec{v})^2 \leq \vec{u}^2 \cdot \vec{v}^2$

$\vec{u} \cdot \vec{v} \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\begin{pmatrix}1-x\\x-y\\y-z\\z\end{pmatrix}$

D'après Cauchy-Schwarz appliqué à $\vec{u}(1, 1, 1, 1)$ et $\vec{v}(1-x, x-y, y-z, z) \in \mathbb{R}^4$.

$\Rightarrow$ On a : $(1\times(1-x) + 1\times(x-y) + 1(y-z) + 1\times z)^2$
$\leq \underbrace{4\times((1-x)^2 + (x+y)^2 + (y-z)^2 + z^2)}$

$\Rightarrow 1 \leq 4\times$

Si ces vecteurs sont colinéaire $1-x = x-y = y-z = z$

càd : $z = \frac{1}{4}$, $y = \frac{1}{2}$ $x = \frac{3}{4}$

$\Rightarrow$ Solution final : $\left(\frac{3}{4} ; \frac{1}{2} ; \frac{1}{4}\right)$

~~donc~~ $y = 2z$, $x = y + z = 3z$ et $z = 1 - x = 1 - 3z \ldots$

$\vec{u}^2 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = 4$

$\vec{v}^2 = \begin{pmatrix}1-x\\x-y\\y-z\\z\end{pmatrix}\begin{pmatrix}1-x\\x-y\\y-z\\z\end{pmatrix} = (1-x)^2 + (x-y)^2 + (y-z)^2 + z^2$.

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Exercice 7.

→ Symétrique

$E = \mathbb{R}_2[X]$

$\langle P, Q\rangle = P(0)Q(0) + P'(0)Q'(0) + P''(0)Q''(0)$

$$M = \begin{pmatrix} \langle 1,1\rangle & \langle 1,X\rangle & \langle 1,X^2\rangle \\ \langle X,1\rangle & \langle X,X\rangle & \langle X,X^2\rangle \\ \langle X^2,1\rangle & \langle X^2,X\rangle & \langle X^2,X^2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$M_1 = 1 > 0 \quad M_2 = 1 > 0 \quad M_3 = 4 > 0$