② ①

TD 2 : Espace vectoriels normés

Exercice 1

$N_1 : \mathbb{R}^2 \to \mathbb{R}$

$(x_1, x_2) \to |x_1| + |x_2|$

* $\forall (x_1, x_2) \in \mathbb{R}^2 \quad |x_1| + |x_2| \geqslant 0$ donc $N_1(x_1, x_2) \in \mathbb{R}_+$

* $\forall (x_1, x_2) \in \mathbb{R}^2 \quad N_1(x_1, x_2) = 0 \Rightarrow |x_1| + |x_2| = 0$

$\Rightarrow |x_1| = |x_2| = 0$

$\Rightarrow (x_1, x_2) = (0, 0)$

* $\forall \lambda \in \mathbb{R} \quad \forall (x_1, x_2) \in \mathbb{R}^2 \quad N_1(\lambda(x_1, x_2)) = N(\lambda x_1, \lambda x_2)$

$= |\lambda x_1| + |\lambda x_2|$

$= |\lambda||x_1| + |\lambda||x_2|$

$= |\lambda||x_1 + x_2|$

$= |\lambda| N_1(x_1, x_2)$

* $\forall ((x_1, x_2), (x'_1, x'_2)) \in (\mathbb{R}^2)^2$

$N_1((x_1, x_2) + (x'_1, x'_2)) = N_1(x_1 + x'_1, x_2 + x'_2)$

$= |x_1 + x'_1| + |x_2 + x'_2| \leqslant |x_1| + |x'_1| + |x_2| + |x'_2|$

$= N_1(x_1, x_2) + N_1(x'_1, x'_2)$

Exercice 2.

$E = \mathbb{R}^2 \quad N : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x,y) \longmapsto \sup\limits_{t\in\mathbb{R}} \dfrac{|tx+y|}{1+t^2}$

1)

* $\lim\limits_{t\to\pm\infty} \dfrac{tx+y}{1+t^2} = 0$ donc $\sup\limits_{t\in\mathbb{R}} \dfrac{|tx+y|}{1+t^2} \in \mathbb{R}^+$

* $\forall (x,y)\in\mathbb{R}^2 \quad N(x,y) = 0 \Rightarrow \sup\limits_{t\in\mathbb{R}} \dfrac{|tx+y|}{1+t^2} = 0$

$\Rightarrow \forall t \in \mathbb{R} \quad tx+y = 0$

$\Rightarrow x = y = 0$

$\Rightarrow (x,y) = (0,0)$ separation : ok.

* $\forall \lambda \in \mathbb{R} \; \forall (x,y)\in\mathbb{R}^2 \; N(\lambda(x,y)) = N(\lambda x, \lambda y) = \sup\limits_{t\in\mathbb{R}} \dfrac{|t\lambda x+\lambda y|}{1+t^2}$

$= \sup\limits_{t\in\mathbb{R}} \dfrac{|\lambda||tx+y|}{1+t^2}$

$= |\lambda| \sup\limits_{t\in\mathbb{R}} \dfrac{|tx+y|}{1+t^2}$

$= |\lambda| N(x,y)$

$\Rightarrow$ homogénéité : ok.

* $\forall (x,y), (x',y') \in (\mathbb{R}^2)^2$

$N((x,y)+(x',y')) = N(x+x', y+y')$

$= \sup\limits_{t\in\mathbb{R}} \dfrac{|t(x+x')+(y+y')|}{1+t^2}$

$\leq \sup\limits_{t\in\mathbb{R}} \dfrac{|tx+y|+|tx'+y'|}{1+t^2}$

$\leq \sup_{t\in\mathbb{R}} \frac{|tx+y|}{1+t^2} + \sup_{t\in\mathbb{R}} \frac{|tx'+y'|}{1+t^2} = N(x,y) + N(x',y')$

$\Rightarrow$ inégalité triangulaire vérifié

2) $\bar{B}(0,1) = \{(x,y) \in \mathbb{R}^2 \ / \ N(x,y) \leq 1\}$

$N(x,y) \leq 1 \iff \sup_{t\in\mathbb{R}} \frac{|tx+y|}{1+t^2} \leq 1$

$\iff \forall t \in \mathbb{R} \quad \frac{|tx+y|}{1+t^2} \leq 1$

$\iff \forall t \in \mathbb{R} \quad -1 \leq \frac{tx+y}{1+t^2} \leq 1$

$\iff \forall t \in \mathbb{R} \quad -1-t^2 \leq tx+y \leq 1+t^2$

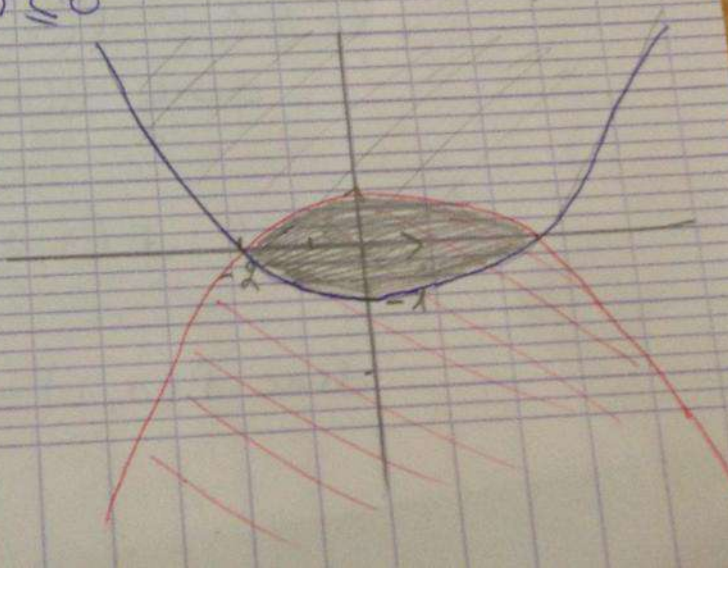
$\iff \forall t \in \mathbb{R} \ -1-t^2 \leq tx+y$ et $tx+y \leq 1+t^2$

$\iff \forall t \in \mathbb{R} \quad \begin{cases} -1-t^2 \leq tx+y \\ tx+y \leq 1+t^2. \end{cases}$

$\iff \forall t \in \mathbb{R} \quad \begin{cases} 0 \leq t^2+xt+(1+y) \\ 0 \leq t^2-xt+(1-y) \end{cases}$

$\iff \begin{cases} x^2-4(1+y) \leq 0 \\ x^2-4(1-y) \leq 0 \end{cases}$

$\iff \begin{cases} \frac{x^2}{4}-1 \leq y \\ y \leq 1-\frac{x^2}{4} \end{cases}$



Exercice 7

$d : E^2 \to \mathbb{R}^+$

$(x,y) \to \begin{cases} 0 \text{ si } x = y \\ 1 \text{ si } x \neq y \end{cases}$

1) • $\forall (x,y) \in E^2 \quad d(x,y) = 0 \Rightarrow x = y$

• $\forall (x,y) \in E^2 \quad d(x,y) \overset{?}{=} d(y,x)$

• si $x = y$ $d(x,y) = 0$ et $d(y,x) = 0$

donc $d(x,y) = d(y,x)$

• si $x \neq y$ $d(x,y) = 1$ et $d(y,x) = 1$

donc $d(x,y) = d(y,x)$

• $\forall (x,y,z) \in E^3 \quad d(x,z) \leq d(x,y) + d(y,z)$

$\to x = y = z \qquad 0 \leq 0 + 0$

$\to x = y \neq z \qquad 1 \leq 0 + 1$

$\to x \neq y = z \qquad 1 \leq 1 + 0$

$\to x = z \neq y \qquad 0 \leq 1 + 1$

$\to x \neq y \neq z \qquad 1 \leq 1 + 1$

2) $\forall r > 0$

$B(a, r) = \{ x \in E \quad d(x,a) < r \}$

• Si $r < 1$ $d(x,a) < 1 \Rightarrow d(x,a) = 0 \Rightarrow x = a$

donc $B(a,r) = \{a\}$.

• Si $r > 1$ $d(x,a) \leq r \Leftrightarrow d(x,a) = 1 \Rightarrow x \neq a$

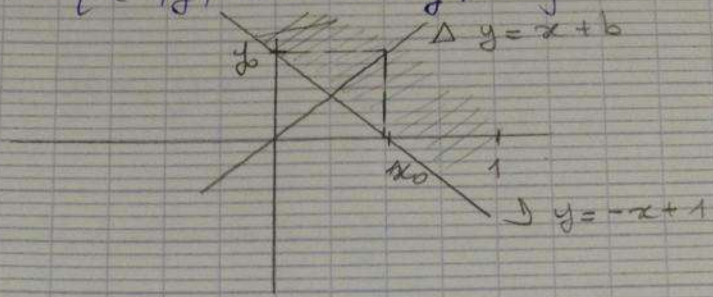
pour $\forall x \in E$

donc $B(a,r) = E$ si $r > 1$.

Exercice 8

$\mathbb{R}^2$

$X = \{(x,y) \in \mathbb{R}^2 \ x+y > 1\}$



$\Delta : \ y = x + b \qquad \Delta : y = x + (y_0 - x_0)$

$y_0 = x_0 + b$

$b = y_0 - x_0$

$$H : \begin{cases} y_H = -x_H + 1 \\ y_H = x_H + (y_0 - x_0) \end{cases}$$

$$\begin{cases} x_H = 1 - y_H = \dfrac{2 - y_0 + x_0 - 1}{2} = \dfrac{x_0 - y_0 + 1}{2} \\ y_H = \dfrac{y_0 - x_0 + 1}{2} \end{cases}$$

$$d(M_0, H) = \sqrt{(x_H - x_M)^2 + (y_H - y_M)^2}$$

$$= \sqrt{\left(\frac{x_0 - y_0 + 1}{2} - x_0\right)^2 + \left(\frac{y_0 - x_0 + 1}{2} - y_0\right)^2}$$

$$= \sqrt{\left(\frac{1 - x_0 - y_0}{2}\right)^2 + \left(\frac{1 - x_0 - y_0}{2}\right)^2}$$

$$= \sqrt{\frac{(1 - x_0 - y_0)^2}{2}}$$

$$= \frac{\sqrt{2}}{2}(1 - x_0 - y_0) = \frac{\sqrt{2}}{2}\left|1 - (x_0 + y_0)\right| > 0$$

On pose $r = \frac{\sqrt{2}}{4}\left(1-(x_0+y_0)\right)$

$B(A_0, r) \subset X$ donc X est un ouvert.