TD3 : Fonctions de deux variable.

Exercice 1

1. $f(x,y) = \frac{x^5 y^3}{x^6 + y^4}$

$\cdot\ f(x,0) = 0 \quad \cdot\ f(0,y) = 0 \quad \cdot\ f(x,x) = \frac{x^8}{x^6+x^4} = \frac{x^4}{x^2+1} \xrightarrow[0]{} 0$

Déci :
$\forall r > 0,\ \forall (x,y) \in B((0,0), r) \qquad \|(x,y)\|_\infty = \sup(|x|, |y|)$

$\rightarrow$ Si $|y| \leq |x|$: $|f(x,y) - 0| = \frac{|x|^5 |y|^3}{x^6+y^4} \leqslant \frac{|x|^5 |x|^3}{x^6} = \frac{x^8}{x^6} = x^2 \leq r^2 \xrightarrow[r \to 0]{} 0$

$x^6 \leq x^6 + y^4$

$\frac{1}{x^6+y^4} \leq \frac{1}{x^6}$.

$\cdot$ Si $|x| \leq |y|$: $|f(x,y) - 0| = \frac{|x|^5|y|^3}{x^6+y^4} \leqslant \frac{|y|^5 |y|^3}{y^4} = \frac{y^8}{y^4} = y^4 \leq r^4 \xrightarrow[r \to 0]{} 0$

2) $f(x,y) = \frac{((x-1)^2 + y^2) \ln((x-1)^2 + y^2)}{|x| + |y|}$

$\cdot\ f(x,0) = \frac{(x-1)^2 \ln(x-1)^2}{|x|} = \frac{2(x-1)^2 \ln|x-1|}{|x|}$ au voisinage de 0.

$= \frac{2(x-1)^2 \ln|1-x|}{|x|}$

$\underset{0}{\sim} \frac{2 \times 1 \times (-x)}{|x|}$

on pose $r = \frac{\sqrt{2}}{4}\left(1-(x_0+y_0)\right)$

$B(M_0, r) \subset X$.

$$= \frac{-2x}{|x|} \begin{cases} \xrightarrow{x>0} -2 \\ \xrightarrow{x<0} 2 \end{cases} \neq \text{ donc pas de limite.}$$

3) En $(0,0)$ $f(x,y) = \frac{1+x+y}{x^2-y^2}$

$f(x,0) = \frac{1+x}{x^2} \longrightarrow \infty$: pas de limite.

4) Soit u une fonction d'un intervalle I dans $\mathbb{R}$ tel que $\forall x \in I$ $u(x) \neq 0$.

Montrer que : $\arctan u + \arctan \frac{1}{u} = \text{signe}(u) \times \frac{\pi}{2}$.

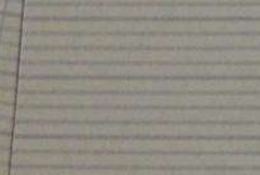
$$\left(\arctan u + \arctan \frac{1}{u}\right)' = \arctan u' + \arctan \frac{1}{u}'$$

$$= \frac{u'}{1+u^2} + \frac{(1/u)'}{1+\left(\frac{1}{u}\right)^2}$$

$$= \frac{u'}{1+u^2} + \frac{\frac{-u'}{u^2}}{\frac{u^2+1}{u^2}} = 0 .$$

Si $u(x) = 1$: $\arctan 1 + \arctan \frac{1}{1} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$.

Si $u(x) = -1$: $\arctan(-1) + \arctan\left(\frac{1}{-1}\right) = -\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$



D'après la question :

$$f(x,y) = \arctan \frac{x}{y} + \arctan \frac{y}{x} = \text{signe}\left(\frac{x}{y}\right) \times \frac{\pi}{2}.$$

=> pas de limite

Exercice 2

• $f(x) = \dfrac{x^2-y^2}{x^2+y^2}$ défini sur $(\mathbb{R}^2)^* = \mathbb{R}^2 \setminus \{0,0\}$

$\neq (\mathbb{R}^*)^2 = \mathcal{A}$

Pour $f(x,0) = \dfrac{x^2}{x^2} = 1$

$f(0,y) = \dfrac{-y^2}{y^2} = -1$

$\left.\right\}$ $f(x,0) \neq f(0,y)$ $\Leftrightarrow$ pas de limite en $(0,0)$

• $g(x,y) = x \arctan \dfrac{y}{x}$

$$D_g = \{ (x,y) \in \mathbb{R} \,/\, x \neq 0 \}$$

$\forall r > 0 \quad \forall (x,y) \in B((0,y_0), r) \quad \|(x,y)\|_\infty = \sup(|x|,|y|)$

$|g(x,y) - 0| = |x| \left|\arctan \dfrac{y}{x}\right| \leq \dfrac{\pi}{2}|x| \leq \dfrac{\pi}{2} r \underset{r \to 0}{\longrightarrow} 0$

$$g : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(x,y) \longrightarrow \begin{cases} x \arctan \dfrac{y}{x} & \text{si } x \neq 0 \\ 0 & \text{si } x = 0 \end{cases}$$

Exercice 3

1_ $\forall t \in \mathbb{R} \quad |\sin t| \leq |t|$

$\Leftrightarrow \forall t \in \mathbb{R}^+ \quad -t \leq \sin t \leq t.$

$\forall t \in \mathbb{R}^+ \quad 0 \leq t - \sin t = \varphi(t)$

$1 - \cos t = \varphi'(t)$

t	0	$+\infty$
φ'	$+$	
φ	0 $\nearrow$	

$\forall t \in \mathbb{R}^+ \quad 0 \leq t + \sin t = \varphi(t)$

$1 + \cos t = \varphi'(t)$

t	0		$+\infty$
φ'		$+$	
φ	0	$\nearrow$	

$\forall t \in \mathbb{R} \qquad t - \frac{t^2}{2} \leq \cos t \leq 1 - \frac{t^2}{2} + \frac{t^2}{24}$

$\forall t \in \mathbb{R} \quad 0 \overset{?}{\leq} \frac{t^2}{2} - 1 + \cos t = \varphi(t)$

$t - \sin t = \varphi'(t)$

$1 - \cos t = \varphi''(t)$

t	$-\infty$	0	$+\infty$
φ''	$+$		$+$
φ'	$\nearrow$	0	$\nearrow$
φ'	$-$		$+$
φ	$\searrow$	0	$\nearrow$

$\forall t \in \mathbb{R} \quad 0 \overset{?}{\leq} \frac{t^2}{24} - \frac{t^2}{2} + 1 - \cos t = \varphi(t)$

$\frac{t^3}{6} - t + \sin t = \varphi'(t)$

$\frac{t^2}{2} - 1 + \cos t = \varphi''(t)$

0	$-\infty$	0	$+\infty$
ψ''	$+$		$+$
φ'	$\nearrow$	0	$\nearrow$
ψ'	$-$	0	$+$
	$\searrow$	0	$\nearrow$

donc tjs ≥ 0,

donc ok.

• $\forall t \in \mathbb{R} \quad \mathrm{ch}(t) \geq 1 + \frac{t^2}{2} \qquad \mathrm{ch}(t) = \frac{e^x + e^{-x}}{2}$

$\forall t \in \mathbb{R} \quad 0 \leq \frac{t^2}{2} + 1 - \mathrm{ch}(t) = \varphi(t)$

$t - \mathrm{sh}\, t = \varphi'(t) \qquad \mathrm{sh}(t) = \frac{e^x - e^{-x}}{2}$

$-\frac{-2 + e^t + e^{-t}}{2} = 1 - \frac{e^t + e^{-t}}{2} = 1 - \mathrm{ch}\, t = \varphi''(t)$

$= -\frac{2e^{t/2}e^{-t/2} + (e^{t/2})^2 + (e^{-t/2})^2}{2}$

$= -\frac{(e^{t/2} + e^{-t/2})^2}{2} \leq 0.$

t	$-\infty$	0	$+\infty$
φ''	$-$		$-$
φ'	$\searrow$	0	$\searrow$
φ'	$+$	0	$-$
φ	$\nearrow$	0	$\searrow$

$f(x,y) = \frac{\sin^3 x}{\mathrm{ch}\, x - \cos y}$

$\underbrace{\mathrm{ch}\, x}_{\geq 1} - \underbrace{\cos y}_{\leq 1} = 0 \iff \mathrm{ch}\, x = \cos y = 1$

$\iff \begin{cases} x = 0 \\ y = 2k\pi \end{cases}$

$\mathrm{Def} = \mathbb{R}^2 \setminus \{(0, 2k\pi)\ k \in \mathbb{Z}\}.$

$-t \leq \sin t \leq t \qquad x^2 \leq |\mathrm{ch}\, x| - |\cos x| \leq |\mathrm{ch}\, x - \cos y|$

$1 + \frac{t^2}{2} \leq \mathrm{ch}\, t$

$1 - \frac{t^2}{2} \leq \cos t \leq 1 - \frac{t^2}{2} + \frac{t^4}{24}$

$$|f(x,y)| = \left| \frac{\mathrm{sh}^{-3} x}{\mathrm{ch}\, x - \cos y} \right| \leq ?$$

Si $|x| \leq |y|$ $\quad |f(x,y)| \leq \dfrac{r^3}{r^2} = r \xrightarrow[r \to 0]{} 0$