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UNIVERSITY

Faculty of Computing, Engineering

and Technology

Mathematical Formulae

For

CE00998-3

Coding and Transformations

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Calculus Properties

Product Rule

If $y = uv$, where u and v are functions of x then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule

If $y = \frac{u}{v}$, where u and v are functions of x then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Chain Rule (or Function of a Function)

If $y = f(g(x))$ and we substitute $t = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Integration by Parts

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$$

Table of Derivatives and Integrals

Function (Integral)	Derivative (Function)
k (constant)	0
x^n	nx^{n-1}
$\frac{x^{n+1}}{n+1}$	$x^n \quad (n \neq -1)$
$\log_e\lvert x\rvert$ or $\ln\lvert x\rvert$	$\frac{1}{x}$
$\ln(ax+b)$	$\frac{a}{ax+b}$
e^{ax}	ae^{ax}
$\sin(x)$	$\cos(x)$
$\sin(ax)$	$a\cos(ax)$
$\cos(x)$	$-\sin(x)$
$\cos(ax)$	$-a\sin(ax)$
$\tan(x)$	$\sec^2(x)$
$\tan(ax)$	$a\sec^2(ax)$

In all the above results the constant of integration has been omitted.

Eulers Formula

$$\cos(\vartheta) = \frac{e^{i\vartheta} + e^{-i\vartheta}}{2} \qquad \sin(\vartheta) = \frac{e^{i\vartheta} - e^{-i\vartheta}}{2i}$$

$$e^{i\vartheta} = \cos(\vartheta) + i\sin(\vartheta) \qquad e^{in\vartheta} = \cos(n\vartheta) + i\sin(n\vartheta)$$

Trigonometric Identities

$$\begin{aligned}\cos(2A) &= \cos^2(A) - \sin^2(A) \\ &= 2\cos^2(A) - 1 \\ &= 1 - 2\sin^2(A)\end{aligned}$$

$$\sin(2A) = 2\cos(A)\sin(A)$$

Fourier Series

1. Whole-range series

For an interval $(0, T)$

Series: $$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{2n\pi x}{T}\right) + b_n \sin\left(\frac{2n\pi x}{T}\right)\right\}$$

Coefficients: $$a_0 = \frac{2}{T}\int_0^T f(x)dx$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2n\pi x}{T}\right)dx \qquad b_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2n\pi x}{T}\right)dx$$

Special case of even function

Series: $$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{T}\right)$$

Coefficients: $$a_0 = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)dx \qquad a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\cos\left(\frac{2n\pi x}{T}\right)dx$$

Special case of odd function

Series: $$\sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{T}\right)$$ Coefficients: $$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\sin\left(\frac{2n\pi x}{T}\right)dx$$

Complex Form of Fourier Series

Series: $$\sum_{n=-\infty}^{\infty} c_n e^{i2n\pi x/T}$$

Coefficients: $$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\, e^{-i2n\pi x/T}dx \qquad n = 0,\ \pm 1, \pm 2, \ldots$$

Fourier Transform

Definition

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad \text{with inverse} \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

For an even function:

$$F(\omega) = 2\int_0^\infty \mathrm{f(t)} \cos(\omega\, t)\, dt$$

and for an odd function :

$$F(\omega) = -2\, i \int_0^\infty f(t) \sin(\omega\, t)\, dt$$

Fourier Transform Properties

1.	Transformation	$f(t) \leftrightarrow F(\omega)$
2.	Linearity	$a_1 f_1(t) + a_2 f_2(t) \leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega)$
3.	Symmetry	$F(t) \leftrightarrow 2\pi\, f(-\omega)$
4.	Scaling	$f(at) \leftrightarrow \frac{1}{\lvert a \rvert} F\left(\frac{\omega}{a}\right)$
5.	Delay	$f(t - t_0) \leftrightarrow e^{-i\, t_0\, \omega} F(\omega)$
6.	Modulation	$e^{i\, \omega_0\, t} f(t) \leftrightarrow F(\omega - \omega_0)$
7.	Convolution	$f_1 * f_2(t) \leftrightarrow F_1(\omega) F_2(\omega)$
8.	Multiplication	$f_1(t) f_2(t) \leftrightarrow \frac{1}{2\pi} F_1 * F_2(\omega)$
9.	Time Differentiation	$\frac{d^n}{dt^n} f(t) \leftrightarrow (i\omega)^n F(\omega)$
10.	Time Integration	$\int f(t) dt \leftrightarrow \frac{F(\omega)}{i\omega}$
11.	Frequency Differentiation	$tf(t) \leftrightarrow i \frac{dF}{d\omega}$
12.	Frequency Integration	$\frac{f(t)}{-it} \leftrightarrow \int F(\omega') d\omega'$
13.	Reversal	$f(-t) \leftrightarrow F(-\omega)$

Note: $\operatorname{sinc}(x) = \frac{\sin(x)}{x}$

Useful Fourier Transforms

	Time Function, $f(t)$	Fourier Transform, $F(\omega)$	
1.	$e^{-at}u(t)$	$\dfrac{1}{a+i\omega}$	$a>0$
2.	$-e^{at}u(-t)$	$\dfrac{1}{i\omega-a}$	$a>0$
3.	$te^{-at}u(t)$	$\left(\dfrac{1}{a+i\omega}\right)^2$	$a>0$
4.	$g_T(t)=\begin{cases}1, & \lvert t\rvert<\dfrac{T}{2}\\ 0, & otherwise\end{cases}$ $=u\left(t+\dfrac{T}{2}\right)-u\left(t-\dfrac{T}{2}\right)$	$T\text{sinc}\left(\dfrac{\omega T}{2}\right)$	
5.	$\begin{cases}A\left(1-\dfrac{\lvert t\rvert}{T}\right) & \lvert t\rvert<T\\ 0 & \lvert t\rvert>T\end{cases}$ *or* $A\left(1-\dfrac{\lvert t\rvert}{T}\right)\left(u(t+T)-u(t-T)\right)$	$AT\text{sinc}^2\left(\dfrac{\omega T}{2}\right)$	
6.	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	
7.	$e^{-at}\sin(\omega_0 t)u(t)$	$\dfrac{\omega_0}{(a+i\omega)^2+\omega_0^2}$	
8.	$e^{-at}\cos(\omega_0 t)u(t)$	$\dfrac{a+i\omega}{(a+i\omega)^2+\omega_0^2}$	
9.	e^{-at^2}	$\sqrt{\dfrac{\pi}{a}}e^{-\frac{\omega^2}{4a}}$	
10.	$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\dfrac{1}{(i\omega+a)^n}$	
11.	$\dfrac{1}{a^2+t^2}$	$\dfrac{\pi}{a}e^{-a\lvert\omega\rvert}$	
12.	$\dfrac{\cos(bt)}{a^2+t^2}$	$\dfrac{\pi}{2a}\left\{e^{-a\lvert\omega-b\rvert}-e^{-a\lvert\omega+b\rvert}\right\}$	
13.	$\dfrac{\sin(bt)}{a^2+t^2}$	$\dfrac{\pi}{2aj}\left\{e^{-a\lvert\omega-b\rvert}-e^{-a\lvert\omega+b\rvert}\right\}$	
14.	$\cos(\omega_0 t)\left\{u\left(t+\dfrac{T}{2}\right)-u\left(t-\dfrac{T}{2}\right)\right\}$	$\dfrac{T}{2}\left\{\text{sinc}\left(\dfrac{(\omega-\omega_0)T}{2}\right)+\text{sinc}\left(\dfrac{(\omega+\omega_0)T}{2}\right)\right\}$	

Useful Fourier Transform (Generalised Functions)

Time Function	Fourier Transform
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$u(t)$	$\pi\delta(\omega)+\frac{1}{i\omega}$
$\mathrm{sgn}(t)=u(t)-u(-t)$	$\frac{2}{i\omega}$
e^{jat}	$2\pi\delta(\omega-a)$
$\sin(at)$	$\frac{\pi}{i}\big(\delta(\omega-a)-\delta(\omega+a)\big)$
$\cos(at)$	$\pi\big(\delta(\omega-a)+\delta(\omega+a)\big)$
$tu(t)$	$i\pi\delta'(\omega)-\frac{1}{\omega^2}$
t^n	$2\pi i^n\delta^{(n)}(\omega)$
$\|t\|$	$-\frac{2}{\omega^2}$
$\delta^{(n)}(t)$	$(i\omega)^n$

Discrete Fourier Transform Formulae

$$F_m=\sum_{n=0}^{N-1} f_n \exp\left(-\frac{i2\pi nm}{N}\right) \quad m=0,1,\ldots,N-1$$

$$f_n=\frac{1}{N}\sum_{k=0}^{N-1} F_k \exp\left(\frac{i2\pi nk}{N}\right) \quad n=0,1,\ldots,N-1$$

Huffman Coding

Average Code Length per symbol:

$$L=\sum_i l_i\ p_i$$

Entropy

$$H(S)=-\sum_i p_i\ \log_2(p_i)$$

$$\log_2(x)=\frac{\ln(x)}{\ln(2)}$$