Théorie du Signal (1)

Convolution et auto-corrélation

Exercice 1 : $\alpha > 0$

Cours autocorrélation :

Energie finie : $\gamma_x(t) = \int x(t)\overline{x(t-\tau)}\,dt$

$\gamma_x(-\tau) = \overline{\gamma_x(\tau)}$

$\gamma_x(0) =$ energie E

$|\gamma_x(\tau)| \leq \gamma_x(0)$

Puissance finie : $\gamma_x(t) = \langle x(t)\overline{x(t-\tau)}\rangle$

$= \lim\limits_{T \to +\infty} \frac{1}{T} \int_{[T]} x(t)\overline{x(t-\tau)}\,dt$

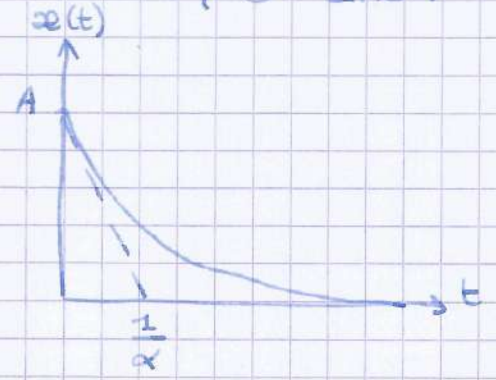
$x(t)$ periodique $\Rightarrow \gamma_x(\tau)$ perio

$\gamma_x(\tau) = \frac{1}{T_0} \int_{[T_0]} x(t)\overline{x(t-\tau)}\,dt$

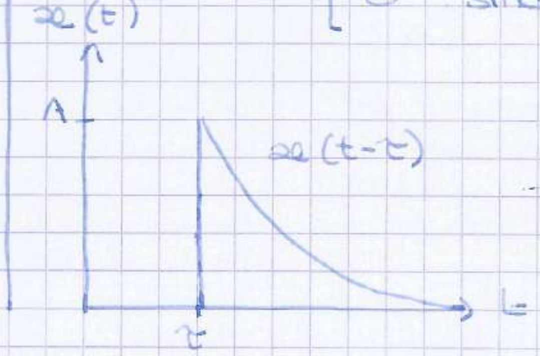
$|\gamma_x(\tau)| \leq \gamma_x(0) = P$

Exponentielle amortie : energie finie

$x(t) = \begin{cases} Ae^{-\alpha t} & \text{si } t \geq 0 \\ 0 & \text{sinon} \end{cases}$



$x(t-\tau) = \begin{cases} Ae^{-\alpha(t-\tau)} & \text{si } t-\tau \geq 0 \\ 0 & \text{sinon} \end{cases}$



$$\gamma_x(\tau) = \int_{\mathbb{R}} x(t)\,x(t-\tau)\,dt = \int_{\tau}^{+\infty} x(t)\,x(t-\tau)\,dt$$

$$= \int_{\tau}^{+\infty} A^2 e^{-\alpha t} e^{-\alpha(t-\tau)}\,dt$$

$$= \int_{\tau}^{+\infty} A^2 e^{-2\alpha t + \alpha\tau}\,dt$$

$$= A^2 \int_{\tau}^{+\infty} e^{-2\alpha t + \alpha\tau}\,dt$$

$$= A^2 e^{\alpha\tau} \int_{\tau}^{+\infty} e^{-2\alpha t}\,dt$$

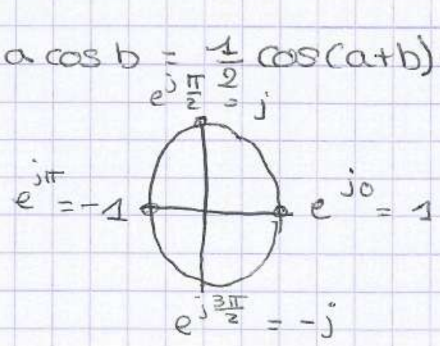
$$= \frac{A^2 e^{\tau}}{-2\alpha}\left[e^{-2\alpha t}\right]_{\tau}^{+\infty}$$

$$= \frac{A^2 e^{\alpha\tau}}{2\alpha} e^{-2\alpha\tau} = \frac{A^2}{2\alpha} e^{-\alpha\tau} \quad (\tau \geq 0)$$

$\tau \leq 0 \qquad \gamma_x(\tau) = \gamma_x(-\tau) = \frac{A^2}{2\alpha} e^{\alpha\tau}$

$\gamma_x(\tau) = \frac{A^2}{2\alpha} e^{-\alpha|\tau|}$

$\cos a \cos b = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$



$|\gamma_x(\tau)| \leq \gamma_x(0) = \frac{A^2}{2\alpha} = E$

Exercice 2 : Signal sinusoïdal : puissance finie

22) $x(t) = A\cos(2\pi f_0 t + \varphi)$

$$\gamma_x(\tau) = \frac{1}{T_0}\int_{[T_0]} x(t)\, x(t-\tau)\, dt$$

$$= \frac{1}{T_0}\int_{[T_0]} A^2\cos(2\pi f_0 t + \varphi)\cos(2\pi f_0(t-\tau)+\varphi)\, dt$$

$$= \frac{A^2}{T_0}\Big[\int_{[T_0]} \frac{1}{2}\cos(2\pi f_0 + \varphi + 2\pi f_0(t-\tau) + \varphi)\, dt$$

$$+ \int_{[T_0]} \frac{1}{2}\cos(2\pi f_0 t - 2\pi f_0(t-\tau))\, dt\Big]$$

$$= \frac{A^2}{2T_0}\int_{[T_0]}\cos[4\pi f_0 t - 2\pi f_0\tau + 2\varphi)\, dt + \int_{[T_0]}\cos(2\pi f_0\tau)\, dt\Big]$$

$$= \frac{A^2}{2T_0}\Big[\underbrace{\int_{T_0}\cos(2\omega_0 t - \omega_0\tau + 2\varphi)\, dt}_{} + \underbrace{\int_{T_0}\cos(\omega_0\tau)\, dt}_{\cos(\omega_0\tau)T_0}\Big]$$

$$= \frac{A^2\cos(\omega_0\tau)}{2} \Rightarrow \max = \frac{A^2}{2}$$

D'où $|\gamma_x(\tau)| \leq \gamma_x(0) = \frac{A^2}{2} = P$

Remarque : $\langle e^{j\omega_0 t}\rangle = \frac{1}{T_0}\int_{T_0} e^{j\omega_0 t}\, dt = \frac{1}{j\omega_0 T_0}\left[e^{j\omega_0 t}\right]_0^{T_0}$

$$= \frac{1}{2j\pi}\left[e^{2j\pi} - 1\right]_0^{T_0}$$

Théorie du signal (2)

Exercice 3 : Convolution par une impulsion

3.1) $\begin{cases} \delta(t) = 0 \quad , \forall t \neq 0 \\ \int_{-\infty}^{+\infty} \delta(t)\, dt = 1 \end{cases}$

(Soit une fonction $x(t) = A\cos(\omega_0 t + \varphi)$)

Rappels : $z(t) = (x * y)(t)$

$$= \int x(\tau)\, y(t-\tau)\, d\tau$$

$$= \int y(\tau)\, x(t-\tau)\, d\tau$$

$z(t) = (x * \delta)(t)$

$$= \int_{\mathbb{R}} x(t-\tau)\, \delta(\tau)\, d\tau$$

$$= \int_{\mathbb{R}} x(t)\, \delta(\tau)\, d\tau$$

$g(u)\,\delta(u) = g(0)\,\delta(u)$

$$= x(t) \int_{\mathbb{R}} \delta(\tau)\, d\tau = \underline{x(t)}$$

3.2) $(x * \delta_{t_0})(t) = \int_{\mathbb{R}} x(t-\tau)\, \delta_{t_0}(\tau)\, d\tau$

$$= \int_{\mathbb{R}} x(t-t_0)\, \delta_{t_0}(\tau)\, d\tau = x(t-t_0) \int_{\mathbb{R}} \delta_{t_0}(\tau)\, d\tau$$

$$= x(t-t_0)$$

$$x_{T_0}(t) = \begin{cases} x(t) \text{ si } t \in \left[-\frac{T_0}{2}, \frac{T_0}{2}\right[\\ 0 \text{ sinon} \end{cases}$$

$x(t) = x_{T_0}(t - T_0)$
$+ x_{T_0}(t - 2T_0) + \dots$
$+ x_{T_0}(t + T_0)$
$+ x_{T_0}(t + 2T_0) + \dots$

3.3)
$$x(t) = \int_{-\infty}^{+\infty} x_{T_0}(t)\, dt$$

$$x(t) = \sum_{n=-\infty}^{+\infty} x_{T_0}(t + nT_0)$$

$$x(t) = x_{T_0}(t) + x_{T_0}(t - T_0) + x_{T_0}(t - 2T_0) + \dots + x_{T_0}(t + T_0) + x_{T_0}(t + 2T_0) + \dots$$

$$= (x_{T_0} * \delta)(t) + (x_{T_0} * \delta_{T_0})(t) + (x_{T_0} * \delta_{2T_0})(t)$$

$$+ \dots + (x_{T_0} * \delta_{-T_0})(t) + (x_{T_0} * \delta_{-2T_0})(t) + \dots$$

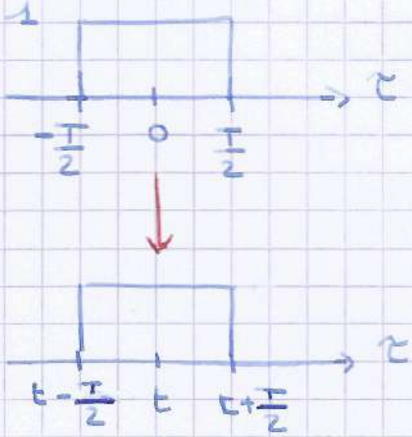
$$= \sum_{n \in \mathbb{Z}} (x_{T_0} * \delta_{nT_0})(t)$$

$$= \left(x_{T_0} * \sum_{n \in \mathbb{Z}} \delta_{nT_0}\right)(t)$$

$$= (x_{T_0} * Ш_{T_0})(t)$$

Théorie du signal (3)

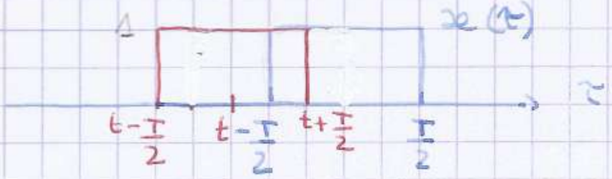
$$z(t) = (x \underset{\text{convolue}}{*} y)(t) = \int_{\mathbb{R}} x(\tau)\, y(t-\tau)\, d\tau$$



1° cas : $t + \frac{T}{2} \leq -\frac{T}{2} \iff t \leq -T$

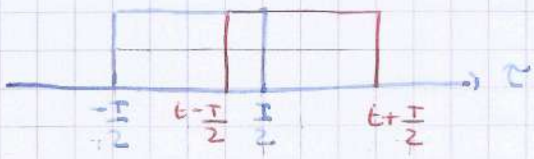
$$z(t) = \int_{\mathbb{R}} x(\tau)\, y(t-\tau)\, d\tau = 0$$

2° cas : $-\frac{T}{2} \leq t + \frac{T}{2} \leq \frac{T}{2} \iff t \in [-T, 0]$



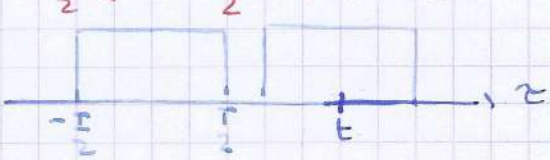
$$z(t) = \int_{\mathbb{R}} x(\tau)\, y(t-\tau)\, d\tau = \int_{-\frac{T}{2}}^{t+\frac{T}{2}} x(\tau)\, y(t-\tau)\, d\tau = \int_{-\frac{T}{2}}^{t+\frac{T}{2}} d\tau = t + \frac{T}{2} - \left(\frac{T}{2}\right) = t + T$$

3° cas : $\frac{T}{2} \leq t + \frac{T}{2} \leq \frac{3T}{2} \iff t \in [0, T]$

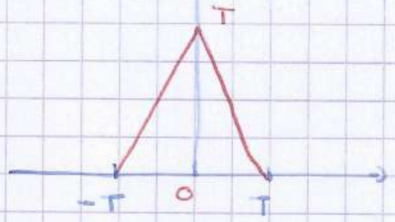


$$z(t) = \int_{t-\frac{T}{2}}^{\frac{T}{2}} d\tau = \frac{T}{2} - \left(t - \frac{T}{2}\right) = T - t$$

4° cas $\frac{T}{2} \leq t - \frac{T}{2} \iff t \geq T$



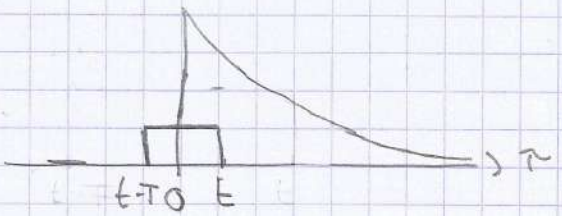
$z(t) = 0$



Exercice 5 :

5.1) $y(t) = (x * h)(t)$

$$= \int x(t-\tau)\, h(\tau)\, d\tau$$



cas 1 : si $t - T \leqslant 0 \Leftrightarrow t > T$

$$y(t) = x(t-\tau)$$

$$=$$

$$\int_0^T x(\tau)\, h(t-\tau)\, d\tau$$

$$\int_0^T A\alpha\, e^{-\alpha(t-\tau)}\, d\tau$$

$$A\alpha \int_0^T e^{-\alpha t + \alpha\tau}\, d\tau$$

$$A\alpha\, e^{-\alpha t} \int_0^T e^{\alpha\tau}$$

$$A e^{-\alpha t} \left[e^{\alpha\tau} \right]_0^T$$

$$A e^{-\alpha t} \left(e^{\alpha T} - 1 \right)$$