THEORIE DU SIGNAL
SUJET 2

Transformée de Fourier

$$X(f) = \int_{\mathbb{R}} x(t)\, e^{-2j\pi f t}\, dt \quad \text{et} \quad x(t) = \int_{\mathbb{R}} X(f)\, e^{2j\pi f t}\, df$$

Exercice 1 :

$$x(t) = \begin{cases} A & \text{si } t \in [-\frac{T}{2}, \frac{T}{2}] \\ 0 & \text{sinon} \end{cases}$$

1.1) $X(f) = \int_{-\frac{T}{2}}^{\frac{T}{2}} A\, e^{-2j\pi f t}\, dt$

$$= A \int_{-T/2}^{T/2} e^{-2j\pi f t}\, dt$$

$$= \frac{-A}{2j\pi f} \left[e^{-2j\pi f t} \right]_{-T/2}^{T/2}$$

$$= -\frac{A}{2j\pi f}\left(e^{-j\pi f T} - e^{j\pi f T} \right) \qquad e^{i\theta} - e^{-i\theta} = 2j\sin\theta$$

$$= \frac{A}{\pi f} \sin(\pi f T)$$

Si $f = 0$ $\quad X(0) = \int_{\mathbb{R}} x(t)\, dt = AT$

$$X(f) = \begin{cases} AT \dfrac{\sin(\pi f T)}{2\pi f T} & \text{si } f \neq 0 \\ AT & \text{sinon} \end{cases} \qquad (= AT \operatorname{sinc}(\pi f T))$$

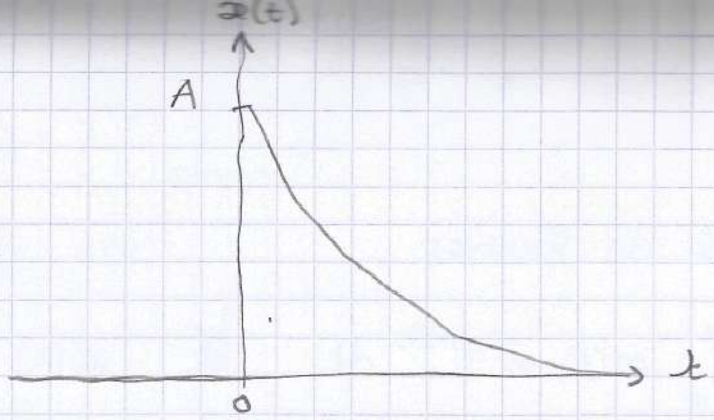
DSE : $\quad S_{xx}(f) = TF\{\gamma_{xx}(\tau)\} = |X(f)|^2$

$$E = \int_{\mathbb{R}} |x(t)|^2\, dt = \int_{\mathbb{R}} |X(f)|^2\, df = \text{Parseval}$$

La DSE s'annule quand $\pi f T$ est un multiple de π donc quand fT est un multiple de 1

$S_{xx}(f) = A^2 T^2 \operatorname{sinc}^2(\pi f T)$

1) $x(t) =$



2) $E = \int_{\mathbb{R}} |x(t)|^2 \, dt \quad = \frac{A^2}{2\alpha}$

3) $X(f) = \int_0^{+\infty} x(t) \, e^{-2j\pi f t} \, dt$

$= \int_0^{+\infty} A e^{-\alpha t} e^{-2j\pi f t} \, dt$

$= A \int_0^{+\infty} e^{-t(\alpha + 2j\pi f)} \, dt$

$= \frac{-A}{\alpha + j\pi f} \left[e^{-t(\alpha + 2j\pi f)} \right]_0^{+\infty}$

$= \frac{A}{\alpha + j\pi f 2} = \frac{A(\alpha - 2j\pi f)}{\alpha^2 - 4\pi^2 f^2} = \frac{A\alpha}{(\alpha^2 - 4\pi^2 f^2)} + j \frac{-2A\pi f}{\alpha^2 - 4\pi^2 f^2}$

$\alpha^2 - 4\pi^2 f^2$

4) $Sx(f) = |X(f)|^2$

$= \sqrt{\frac{A^2\alpha^2}{(\alpha^2 - 4\pi^2 f^2)^2} + \frac{4A^2\pi^2}{(\alpha^2 - 4\pi^2 f^2)}}$

$= \frac{A^2}{\alpha^2 + 4\pi^2 f^2}$

Exercice 3

1) $x(t-t_0) = y(t)$

$Y(f) = \int_{\mathbb{R}} y(t) e^{-2j\pi f t} dt$

$= \int_{\mathbb{R}} x(t-t_0) e^{-2j\pi f t} dt$ $\qquad t - t_0 = t'$

$= \int_{\mathbb{R}} x(t') e^{-2j\pi f (t'+t_0)} dt'$ $\qquad t = t' + t_0$ $\qquad dt' = dt$

$= X(f) e^{-2j\pi f t_0}$

$|Y(f)| = |X(f)|$ $\quad \arg Y(f) = \arg X(f) - 2j\pi f t_0$

2) $Y(f) = \int_{\mathbb{R}} y(t) e^{-2j\pi f t} dt$

$= \int_{\mathbb{R}} x(t) e^{-2j\pi t (f - f_0)} dt$ $\qquad e^{i\theta} \times e^{-i\theta}$

$= X(f - f_0)$

3) $x(t) \cos(2\pi f_0 t) = \frac{1}{2} x(t) e^{2j\pi f_0 t} + \frac{1}{2} x(t) e^{-2j\pi f_0 t} \xrightarrow{TF}$

$\frac{1}{2} X(f-f_0) + \frac{1}{2} X(f+f_0)$

$\delta(f) \xrightarrow{TF'} 1$

$\delta(t) \xrightarrow{TF} 1$

$A e^{j\phi} e^{j2\pi f_0 t} \longrightarrow A e^{j\phi} \delta(f - f_0)$

$A \cos(2\pi f_0 t + \phi) \longrightarrow \frac{A e^{j\phi}}{2} \delta(f - f_0) + \frac{A e^{-j\phi}}{2} \delta(f + f_0)$

$Y(f) = 2\pi j f \; X(f)$

$y(t) = \int_{\mathbb{R}} X(f) \, 2j\pi f \, e^{2j\pi f t} df$

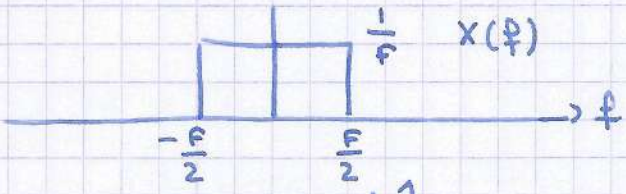
$= \int_{\mathbb{R}} \frac{\partial}{\partial t} \left(X(f) e^{2j\pi f t} \right) df$

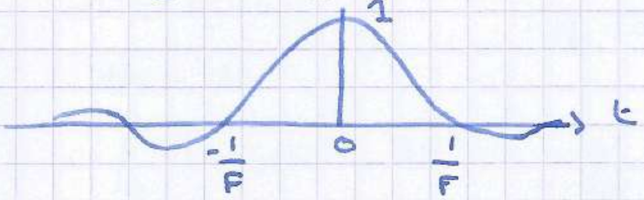
$= \dot{x}(t)$

$\dot{x}(t) \longrightarrow 2j\pi f \; X(f)$.

Exercice 4 :

$$x(t) \xrightarrow{TF} X(f) = \begin{cases} \frac{1}{F} & \text{si } -\frac{F}{2} \leq f \leq \frac{F}{2} \\ 0 & \text{sinon} \end{cases}$$





① $X(f) \xrightarrow{TF^{-1}} x(t) = \text{sinc}(\pi F t)$

② formule de la transformée de Fourrier :

$$X(f) = \int x(t) e^{-2j\pi f t} dt$$

Donc $\int_{\mathbb{R}} x(t)\, dt = X(0) = \frac{1}{F}$

③ $$S_x(f) = |X(f)|^2 = \begin{cases} \frac{1}{F^2} & \text{si } |f| < \frac{F}{2} \\ 0 & \text{sinon} \end{cases}$$

$E = \int S_x(f)\, df$ (PER

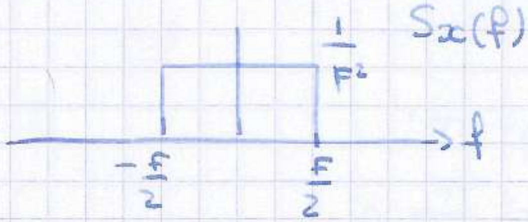
$= \int_{-\frac{F}{2}}^{\frac{F}{2}} \frac{1}{F^2}\, df$

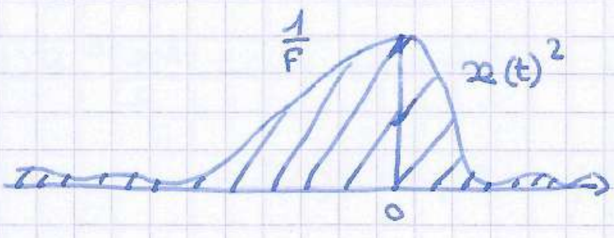
$= \frac{1}{F^2} \int_{-\frac{F}{2}}^{\frac{F}{2}} df$

$= \frac{1}{F^2} \left[f \right]_{-\frac{F}{2}}^{\frac{F}{2}}$

$= \frac{1}{F^2} \left[\frac{F}{2} + \frac{F}{2} \right]$

$= \frac{1}{F}$





④ $\gamma_x(\tau) = \int_{\mathbb{R}} x(t)\, x(t-\tau)\, dt$

$= TF^{-1}\{ S_x(f) \} = \frac{1}{F} \text{sinc}(\pi F \tau)$

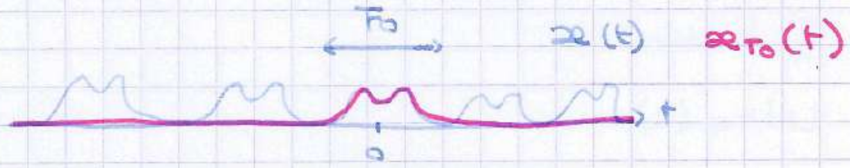
Spectre d'un signal periodique :

↳ Décomposition en série de Fourier

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{2j\pi n f_0 t}$$ (→ on peut l'associer à la transformée de fourier inverse)

$$C_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cdot e^{-2j\pi n f_0 t} \, dt$$ (→ on peut l'associer à une transformée de Fourrier)

$$= \frac{1}{T_0} \int_{\mathbb{R}} x_{T_0}(t) \, e^{-2j\pi n f_0 t} \, dt = \frac{1}{T_0} X_{T_0}(n f_0)$$



$\delta(f - n f_0)$ $\qquad e^{2j\pi f_1 t} \longrightarrow \delta(f - f_1)$

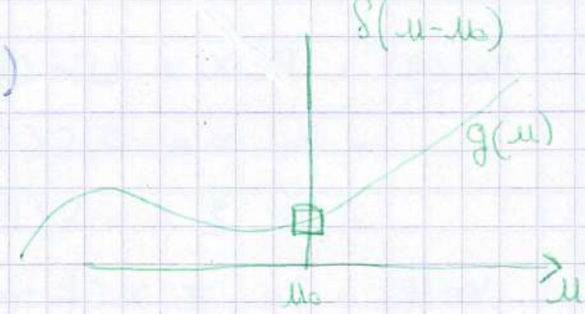
$$\boxed{X(f) = \sum c_n \, \delta(f - n f_0)}$$

$$X(f) = \frac{1}{T_0} \sum_n X_{T_0}(n f_0) \, \delta(f - n f_0)$$

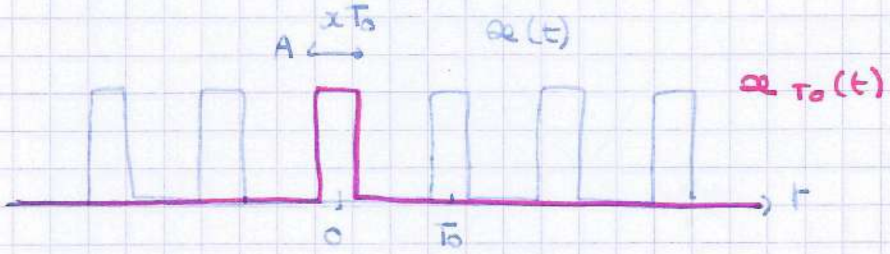
$$= \frac{1}{T_0} \sum_n X_{T_0}(f) \, \delta(f - n f_0)$$

$$= \frac{1}{T_0} X_{T_0}(f) \sum_n \delta(f - n f_0)$$

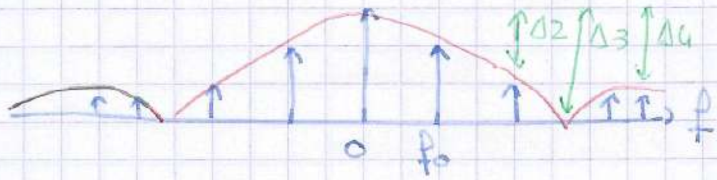
$$= \frac{1}{T_0} X_{T_0}(f) \, Ш_{f_0}(f)$$



$$g(u) \times \delta(u - u_0) = g(u_0) \times \delta(u - u_0)$$



$|X(f)|$ $\qquad \frac{1}{T_0} |X_{T_0}(f)|$

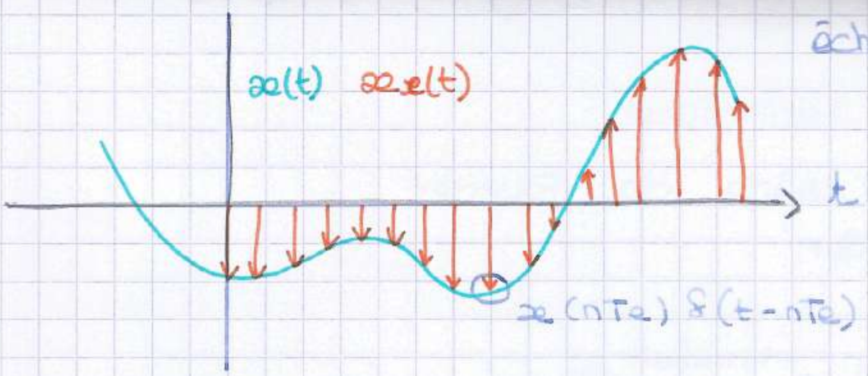


$$X(f) = \sum c_n \, \delta(f - n f_0) \qquad \Delta_n = 20 \log_{10} \left| \frac{c_1}{c_n} \right|$$

$$S_x(f) = \sum |c_n|^2 \, \delta(f - n f_0)$$

$$P = \int S_x(f) \, df = \sum |c_n|^2$$

échantillonné à $f_e = \frac{1}{T_e}$



$$x_e(t) = \sum_n x(nT_e)\,\delta(t - nT_e)$$

$$= \sum_n x(t)\,\delta(t - nT_e)$$

$$= x(t) \sum_n \delta(t - nT_e)$$

$$\boxed{x_e(t) = x(t)\, Ш_{T_e}(t)}$$

$x(t) \longrightarrow X(f)$

$Ш_{T_e}(t) \longrightarrow f_e\, Ш_{f_e}(f)$

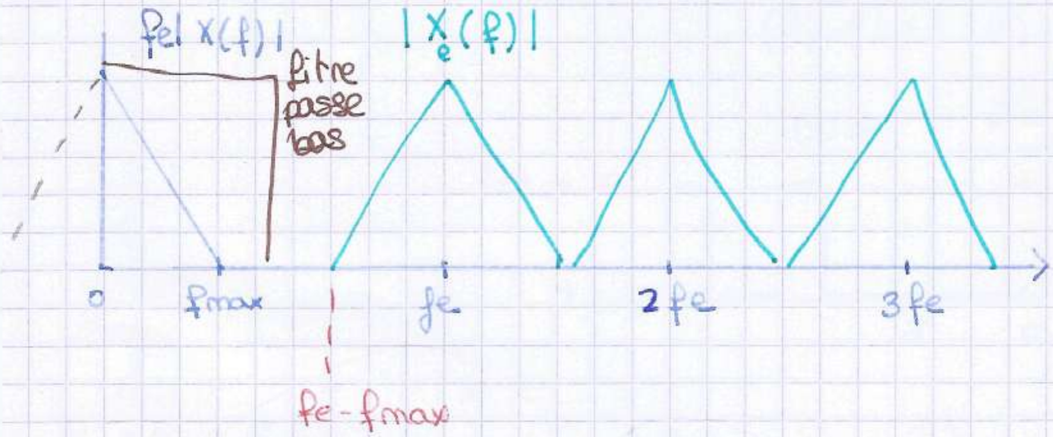
$$X_e(f) = f_e\, X * Ш_{f_e}(f)$$

$$= f_e \cdot X * \sum \delta(f - nf_e)$$

$$= f_e \sum X * \delta(f - nf_e)$$

$$\boxed{X_e(f) = f_e \sum_n X(f - nf_e)}$$

$X(f - f_e)$



$f_{max} < f_e - f_{max}$

$$\boxed{f_e > 2 f_{max}}$$