

3.8.2 Invasion par résolution du système associé

$$A \in M_n(K) \quad \text{mit} \quad Y_{mn} = A_{m,n} X_{n,p}$$

Nun $A^{-1}y = A^{-1} \cdot Ax = I \cdot x = x$

$$\text{ex: } \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{d'on} \quad \begin{cases} x_1 - x_2 = y_1 \\ 2x_1 + 3x_2 = y_2 \end{cases} \quad \times 3 \Rightarrow \begin{cases} 5x_1 = 3y_1 + y_2 \\ x_1 = \frac{3}{5}y_1 + \frac{1}{5}y_2 \end{cases}$$

$$\text{or } 5x_2 = -2y_1 + y_2 \Rightarrow x_2 = -\frac{2}{5}y_1 + \frac{1}{5}y_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3/5 & 1/5 \\ -4/5 & 1/5 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

3.8.3 Méthode "naïve" du pivot de Gauss

On fait le pivot sur les lignes

$$\Rightarrow \begin{array}{cc} \begin{array}{c} A \\ \left(\begin{array}{cc} 1 & -1 \\ 2 & 3 \end{array} \right) \end{array} & \begin{array}{c} I \\ \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \end{array} \end{array} \Rightarrow \begin{array}{cc} \begin{array}{c} A \\ \left(\begin{array}{cc} 1 & -1 \\ 0 & 5 \end{array} \right) \end{array} & \begin{array}{c} I \\ \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \end{array} \end{array} \begin{array}{l} L_1 \\ L_2 - 2L_1 \end{array} \begin{array}{c} I \\ \left(\begin{array}{cc} 1 & 0 \\ -2 & 1 \end{array} \right) \end{array}$$

$$m_1: \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 5 \\ -1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 3 \\ 0 & 1 & 1 \end{pmatrix} \begin{matrix} L_1 \\ L_2 - 3L_1 \\ L_3 + L_1 \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & L_1 & 1 & 0 & 0 \\ 0 & 3 & 3 & L_2 & -2 & 1 & 0 \\ 0 & 0 & 4 & L_3 & \frac{1}{3} & \frac{1}{3} & 1 \end{array} \right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & L_1 - \frac{1}{4} & \frac{7}{12} & \frac{1}{12} & -\frac{1}{4} \\ 0 & 3 & 0 & L_2 - \frac{3}{4} & \frac{13}{4} & \frac{5}{4} & -\frac{3}{4} \\ 0 & 0 & 4 & L_3 & \frac{5}{3} & -\frac{1}{3} & 1 \end{array} \right)$$

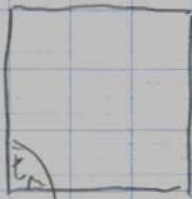
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{L_1} \begin{pmatrix} 7/12 & 1/12 & -3/12 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{L_2/3} \begin{pmatrix} 7/12 & 1/12 & -3/12 \\ -1/12 & 5/12 & -3/12 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{L_3/4} \begin{pmatrix} 7/12 & 1/12 & -3/12 \\ -1/12 & 5/12 & -3/12 \\ 5/12 & -1/12 & 3/12 \end{pmatrix} \leftarrow A^{-1}$$

vérif:

$$\begin{pmatrix} 2 & 1 & -3 \\ -13 & 5 & -3 \\ 5 & -1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 5 \\ -1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix} \times 1/12$$

3.9 Changement de base



$$B = \{\vec{e}_1, \dots, \vec{e}_n\}$$

ancienne

$$B' = \{\vec{e}'_1, \dots, \vec{e}'_n\}$$

nouvelle

$$\vec{e}'_j = \sum_{i=1}^n \alpha_{ij} \vec{e}_i$$

$$P = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix}$$

P est la matrice de passage

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad X' = \begin{pmatrix} x'_1 \\ \vdots \\ x'_m \end{pmatrix}$$

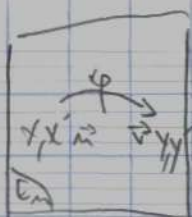
$$\vec{u} = x_1 \vec{e}_1 + \dots + x_n \vec{e}_n = x'_1 \vec{e}'_1 + \dots + x'_m \vec{e}'_m$$

$$\vec{u} = x'_1 (\alpha_{11} \vec{e}_1 + \dots + \alpha_{n1} \vec{e}_n) + \dots + x'_m (\alpha_{1m} \vec{e}_1 + \dots + \alpha_{nm} \vec{e}_n)$$

$$\left. \begin{aligned} x_1 &= \alpha_{11} x'_1 + \dots + \alpha_{n1} x'_m \\ &\vdots \\ x_n &= \alpha_{1n} x'_1 + \dots + \alpha_{nm} x'_m \end{aligned} \right\}$$

$$X = P X'$$

$$\Leftrightarrow X' = P^{-1} X$$



$$X = PX'$$

$$Y = PY'$$

$$Y = AX$$

$$Y' = A'X'$$

avec

$$A = \text{mat}(\varphi; \mathcal{B})$$

$$A' = \text{mat}(\varphi; \mathcal{B}')$$

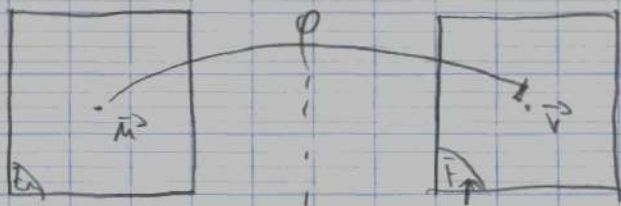
$$A'X' = A'P^{-1}X = Y' = P^{-1}Y = P^{-1}AX$$

$$A'P^{-1} = P^{-1}A \quad A' \Leftrightarrow A'P^{-1}P = P^{-1}AP$$

$$A' = P^{-1}AP$$

On cherche E' tel que A' soit "simple", par ex. $A' = D$ (diagonale)
 alors $D = P^{-1}AP$

$$PD = AP \Rightarrow PDP^{-1} = A$$



base $E_1 \xrightarrow{P} E_2$
 $\vec{x}_1 \rightarrow X_1 \quad X_2$

base $F_1 \xrightarrow{Q} F_2$
 $\vec{y}_1 \rightarrow Y_1 \quad Y_2$

$$A = \text{mat}(\varphi; E_1; F_1)$$

$$B = \text{mat}(\varphi; E_2; F_2)$$

$$Y_1 = AX_1 = QY_2$$

$$Y_2 = BX_2$$

d'où

$$B = Q^{-1}AP$$

Ch IV Déterminants

4.1 Groupe symétrique

$$n \in \mathbb{N}^* \quad \varphi = \begin{pmatrix} 1 & 2 & \dots & n \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \end{pmatrix}$$

$$\text{avec } \alpha_1 = \varphi(1) \dots \alpha_n = \varphi(n) \quad \alpha_i \in [1, \dots, n] \\ \mathbb{N}_m^*$$

φ est bijective

$$\mathfrak{S}_n = \{ \varphi \text{ sur } \mathbb{N}_n^* \}$$

$$Id = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}$$

• composition de permutation

$$\text{ex: } \varphi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \varphi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \varphi_2 \circ \varphi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$\{ \mathfrak{S}_n, \circ \}$ est un groupe non commutatif.

$$\text{transposit: } \tau_{ij} = \begin{pmatrix} 1 & \dots & i & \dots & j & \dots & n \\ 1 & \dots & j & \dots & i & \dots & n \end{pmatrix} \quad \text{ex: } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix} = \tau_{24}$$

$$(\tau_{ij})^{-1} = \tau_{ij} = \tau_{ji}$$

La transposit^{on} est involutive

$$\tau_{ij} \circ \tau_{ij} = Id$$

Toute permutation est le produit de transposition.

ex: $\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 3 & 6 & 1 \end{pmatrix} = \tau_{12} = \tau_{25} = \tau_{56} = \tau_{34}$

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 5 & 6 \end{pmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$

orbite de 1 orbite de 3

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 3 & 5 & 6 \\ 1 & 2 & 4 & 3 & 6 & 5 \\ 1 & 5 & 4 & 3 & 6 & 2 \\ 2 & 5 & 4 & 3 & 6 & 1 \end{pmatrix} \begin{matrix} \tau_{34} \\ \tau_{56} \\ \tau_{56} \\ \tau_{51} \\ \tau_{12} \end{matrix}$$

inversion $\varphi = \begin{pmatrix} 1 & 2 & i & j & \dots & m \\ \alpha_1 & \alpha_2 & \alpha_i & \alpha_j & \dots & \alpha_m \end{pmatrix} \quad (i, j) \text{ petite inversion}$

si $i < j$ et $\alpha_i > \alpha_j$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 3 & 6 & 1 \end{pmatrix}$$

2	petite	1 inversion
5	"	3 "
4	"	2 "
3	"	1 "
6	"	1 "

$I(\varphi) = 8 \text{ inversions}$

On appelle signature de φ , notée $\varepsilon(\varphi) = (-1)^I$

on a vu que φ est un produit de $h=4$ transpositions

$$(-1)^I = (-1)^h$$

$(2 \ 5 \ 4 \ 3 \ 6 \ 1)$ est mis dans le "bon ordre" avec h transpositions et h ayant la parité de I .

4.2 Calcul théorique d'un déterminant

Soit E de dimension n et n vecteurs de E : $\vec{v}_1, \dots, \vec{v}_n$

base $(\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n)$ de E

$$\vec{v}_i = \sum_{j=1}^n a_{ij} \vec{e}_j$$

Soit f une fonction multilinéaire alternée

appliquée E^n
dans K

linéaire / $\vec{v}_i$
 $\forall i$

si $\vec{v}_i$ et $\vec{v}_j$ permutent,
la fonction change de signe

$$\Delta = f(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n) = f\left(\sum_{j_1=1}^n a_{1j_1} \vec{e}_{j_1}, \sum_{j_2=1}^n a_{2j_2} \vec{e}_{j_2}, \dots, \sum_{j_n=1}^n a_{nj_n} \vec{e}_{j_n}\right)$$

$$\Delta = \sum_{j_1=1}^n \sum_{j_2=1}^n \dots \sum_{j_n=1}^n f(\vec{e}_{j_1}, \vec{e}_{j_2}, \dots, \vec{e}_{j_n}) a_{1j_1} \dots a_{nj_n}$$

or f est alternée: $f(\dots, \vec{v}_i, \dots, \vec{v}_j, \dots) = -f(\dots, \vec{v}_j, \dots, \vec{v}_i, \dots)$

$$f(\dots, \vec{v}_i, \dots, \vec{v}_i, \dots) = 0$$

$$\Delta = \sum_{\varphi \in \mathcal{S}_n} a_{1\varphi_1} \dots a_{n\varphi_n} f(\vec{e}_{\varphi_1}, \dots, \vec{e}_{\varphi_n})$$

d'où
$$\Delta = \sum_{\varphi \in \mathcal{S}_n} \varepsilon(\varphi) a_{1\varphi_1} \dots a_{n\varphi_n} f(\vec{e}_1, \dots, \vec{e}_n)$$

Le déterminant Δ de n vecteurs $\vec{v}_1, \dots, \vec{v}_n$ est la fonction multilinéaire alternée de ces n vecteurs prenant la valeur 1 sur la base

cette fonction est unique

$$\Delta = \sum_{\varphi \in \mathcal{S}_n} \varepsilon(\varphi) a_{1\varphi_1} \dots a_{n\varphi_n}$$

4.3 Determinants simples

$$n=1 \quad \vec{v}_1 = a_1 \vec{e}_1$$

$$\Delta_1 = |a_1|$$

$$\Delta_1 = a_1$$

$$\mathcal{B}_1 \quad \varphi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Sigma(\varphi_1) = +1$$

$$A_1 = [a_1] \quad |A_1| = |a_1| = a_1$$

$$n=2 \quad \vec{v}_1 = a_{11} \vec{e}_1 + a_{12} \vec{e}_2$$

$$\vec{v}_2 = a_{21} \vec{e}_1 + a_{22} \vec{e}_2$$

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \det(\vec{v}_1, \vec{v}_2) = \sum_{\varphi \in \mathcal{B}_2} \Sigma(\varphi) a_{1\varphi(1)} a_{2\varphi(2)}$$

$$\mathcal{B}_2 \quad \varphi_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\Sigma(\varphi_1) = +1$$

$$\varphi_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\Sigma(\varphi_2) = (-1)^1 = -1$$

$$\Delta_2 = \Sigma(\varphi_1) a_{1\varphi_1(1)} a_{2\varphi_1(2)} + \Sigma(\varphi_2) a_{1\varphi_2(1)} a_{2\varphi_2(2)}$$

$$\Delta_2 = + a_{11} a_{22} - a_{12} a_{21}$$

$$\left[A_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a \text{ is determinant} \quad |A_2| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \right] \quad \heartsuit$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = -2$$

$$n=3$$

$$\begin{aligned} \vec{v}_1 &= a_{11} \vec{e}_1 + a_{12} \vec{e}_2 + a_{13} \vec{e}_3 \\ \vec{v}_2 &= a_{21} \vec{e}_1 + a_{22} \vec{e}_2 + a_{23} \vec{e}_3 \\ \vec{v}_3 &= a_{31} \vec{e}_1 + a_{32} \vec{e}_2 + a_{33} \vec{e}_3 \end{aligned}$$

$$A_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\sum_3 \varphi_1 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \varphi_2 \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \varphi_3 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \varphi_4 \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \varphi_5 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \varphi_6 \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\varepsilon_1 = +1 \quad \varepsilon_2 = -1 \quad \varepsilon_3 = -1 \quad \varepsilon_4 = +1 \quad \varepsilon_5 = +1 \quad \varepsilon_6 = -1$$

$$\Delta_3 = a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}$$

