

TAI Maths du Réel

$$F(x, y, z) = -6x^2 + 9y^2 + 3xy - 7xz + 2yz + 6x - y + 2z$$

Exercice 1

$$G : \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$(x, y, z) \rightarrow G(x, y, z) = \sqrt{xyz} * \frac{F(x, y, z)}{x^2 + y^2 + z^2} \text{ si } (x, y, z) \neq (0, 0, 0)$$

0 sinon

Soit  $B_{((0,0,0),r)}$  une boule ouverte de centre  $(0, 0, 0)$  et de rayon  $r$

$$\text{Et } \|(x, y, z)\|_\infty = \sup(|x|, |y|, |z|)$$

$$\text{Si } \sup(|x|, |y|, |z|) = |x|$$

$$|x|^3 \geq |x| |y| |z|$$

$$\sqrt{|x|^3} \geq \sqrt{|x| |y| |z|}$$

$$x^2 \leq x^2 + y^2 + z^2$$

$$\frac{1}{x^2} \geq \frac{1}{x^2 + y^2 + z^2}$$

$$-6|x|^2 + 9|x|^2 + 3|x|^2 - 7|x|^2 + 2|x|^2 + 6|x| - |x| + 2|x| \geq F(x, y, z)$$

$$|x|^2 + 7|x| \geq F(x, y, z)$$

$$\sqrt{xyz} * \frac{F(x, y, z)}{x^2 + y^2 + z^2} \leq \sqrt{|x|^3} * \frac{|x|^2 + 7|x|}{x^2}$$

$$\sqrt{xyz} * \frac{F(x, y, z)}{x^2 + y^2 + z^2} \leq \frac{|x|^{\frac{7}{2}} + 7|x|^{\frac{5}{2}}}{x^2}$$

$$\sqrt{xyz} * \frac{F(x, y, z)}{x^2 + y^2 + z^2} \leq |x|^{\frac{3}{2}} \leq r^{\frac{3}{2}} \rightarrow 0 \text{ quand } r \rightarrow 0$$

Le calcul et le résultat sont similaires pour  $\sup(|x|, |y|, |z|) = |y|$  et pour  $\sup(|x|, |y|, |z|) = |z|$ .

Ainsi, la limite en 0 de  $G(x, y, z) = 0$ , donc  $G(x, y, z)$  est continue en 0.

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$$H : \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$(x, y, z) \rightarrow H(x, y, z) = \frac{F(x, y, z)}{x^2 + y^2 + z^2} \text{ si } (x, y, z) \neq (0, 0, 0)$$

0 sinon

$$\text{En } (x, 0, 0) : H(x, 0, 0) = \frac{-6x^2 + 6x}{x^2}. \text{ Quand } x \rightarrow 0, H(x, 0, 0) \rightarrow +\infty$$

$$\text{En } (0, y, 0) : H(0, y, 0) = \frac{9y^2 - y}{y^2}. \text{ Quand } y \rightarrow 0, H(0, y, 0) \rightarrow -\infty$$

$$\text{En } (0, 0, z) : H(0, 0, z) = \frac{2z}{z^2}. \text{ Quand } z \rightarrow 0, H(0, 0, z) \rightarrow +\infty$$

Comme les limites sont infinies et différentes de 0, on peut conclure que la fonction  $H(x, y, z)$  n'est pas continue en 0.

Exercice 3

$$K : \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$(x, y, z) \rightarrow K(x, y, z) = x^3 + y^3 + z^3 + F(x, y, z)$$

$$\frac{\partial K}{\partial x} = 3x^2 - 12x + 3y - 7z + 6$$

$$\frac{\partial K}{\partial y} = 3y^2 + 18y + 3x + 2z - 1$$

$$\frac{\partial K}{\partial z} = 3z^2 - 7x + 2y + 2$$

$$\text{grad } \vec{K} = 0$$

$$\begin{cases} 3x^2 - 12x + 3y - 7z + 6 = 0 \\ 3y^2 + 18y + 3x + 2z - 1 = 0 \\ 3z^2 - 7x + 2y + 2 = 0 \end{cases}$$

$$A(x = -1.17712, y = -6.1554, z = 0.830859)$$

$$B(x = 0.0462651, y = -6.24857, z = -1.89921)$$

$$C(x = 0.323058, y = -0.0354808, z = 0.332851)$$

$$D(x = 2.79333, y = -5.93734, z = -3.13199)$$

$$E(x = 5.51842, y = -1.82872, z = 3.66453)$$

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$$\frac{\partial K}{\partial x^2} = 6x - 12$$

$$\frac{\partial K}{\partial y \partial x} = 3$$

$$\frac{\partial K}{\partial z \partial x} = -7$$

$$\frac{\partial K}{\partial x \partial y} = 3$$

$$\frac{\partial K}{\partial y^2} = 6y + 18$$

$$\frac{\partial K}{\partial z \partial y} = 2$$

$$\frac{\partial K}{\partial x \partial z} = -7$$

$$\frac{\partial K}{\partial y \partial z} = 2$$

$$\frac{\partial K}{\partial z^2} = 6z$$

$$H_k(x, y, z) = \begin{pmatrix} 6x - 12 & 3 & -7 \\ 3 & 6y + 18 & 2 \\ -7 & 2 & 6z \end{pmatrix}$$

En A ( $x = -1.17712$ ,  $y = -6.1554$ ,  $z = 0.830859$ )

$$H_k(x = -1.17712, y = -6.1554, z = 0.830859) = \begin{pmatrix} -19.06272 & 3 & -7 \\ 3 & -18.9324 & 2 \\ -7 & 2 & 4.985154 \end{pmatrix}$$

$$H_1 < 0$$

$$H_2 > 0$$

$$H_3 > 0$$

Donc le point critique n'est pas un extremum de  $K(x, y, z)$ .

En B ( $x = 0.0462651$ ,  $y = -6.24857$ ,  $z = -1.89921$ )

$$H_k(x = 0.0462651, y = -6.24857, z = -1.89921) = \begin{pmatrix} -11.7224094 & 3 & -7 \\ 3 & -19.49142 & 2 \\ -7 & 2 & -11.21524 \end{pmatrix}$$

$$H_1 < 0$$

$$H_2 > 0$$

$$H_3 < 0$$

On a donc un maximum local en B ( $x = 0.0462651$ ,  $y = -6.24857$ ,  $z = -1.89921$ ) de valeur  $K(x = 0.0462651, y = -6.24857, z = -1.89921) = 126.7755983$ .

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En C ( $x = 0.323058$ ,  $y = -0.0354808$ ,  $z = 0.332851$ )

$$H_k(x = 0.323058, y = -0.0354808, z = 0.332851) = \begin{pmatrix} -10.061651 & 3 & -7 \\ 3 & 17.7871152 & 2 \\ -7 & 2 & 1.997106 \end{pmatrix}$$

$$H_1 < 0$$

$$H_2 < 0$$

Donc le point critique n'est pas un extremum de  $K(x, y, z)$ .

En D ( $x = 2.79333$ ,  $y = -5.93734$ ,  $z = -3.13199$ )

$$H_k(x = 2.79333, y = -5.93734, z = -3.13199) = \begin{pmatrix} 4.75998 & 3 & -7 \\ 3 & -53.6204 & 2 \\ -7 & 2 & -18.79194 \end{pmatrix}$$

$$H_1 > 0$$

$$H_2 < 0$$

Donc le point critique n'est pas un extremum de  $K(x, y, z)$ .

En E ( $x = 5.51842$ ,  $y = -1.82872$ ,  $z = 3.66453$ )

$$H_k(x = 5.51842, y = -1.82872, z = 3.66453) = \begin{pmatrix} 21.11052 & 3 & -7 \\ 3 & 7.02768 & 2 \\ -7 & 2 & 21.9818 \end{pmatrix}$$

$$H_1 > 0$$

$$H_2 > 0$$

$$H_3 > 0$$

On a donc un minimum local en E ( $x = 5.51842$ ,  $y = -1.82872$ ,  $z = 3.66453$ ) de valeur  $K(x = 5.51842, y = -1.82872, z = 3.66453) = -84.43939636$ .

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$$x + y + z = 1 \Leftrightarrow x + y + z - 1 = 0$$

$$\text{Posons } g(x, y, z) = x + y + z - 1$$

$$L(x, y, z, \lambda) = F(x, y, z) - \lambda * g(x, y, z)$$

$$L(x, y, z, \lambda) = -6x^2 + 9y^2 + 3xy - 7x^2 + 2yz + 6x - y + 2z - \lambda x - \lambda y - \lambda z + \lambda$$

$$\frac{\partial L}{\partial x} = -12x + 3y - 7z + 6 - \lambda$$

$$\frac{\partial L}{\partial y} = 18y + 3x + 2z - 1 - \lambda$$

$$\frac{\partial L}{\partial z} = -7x + 2y + 2 - \lambda$$

$$\text{grad } \vec{L} = 0$$

$$\begin{cases} -12x + 3y - 7z + 6 - \lambda = 0 \\ 18y + 3x + 2z - 1 - \lambda = 0 \\ -7x + 2y + 2 - \lambda = 0 \\ x + y + z - 1 = 0 \end{cases}$$

$$x = 1 - y - z$$

$$-12(1 - y - z) + 3y - 7z + 6 - \lambda = 0$$

$$-12 + 12y + 12z + 3y - 7z + 6 - \lambda = 0$$

$$15y + 5z - 6 - \lambda = 0 \quad (1)$$

$$18y + 3(1 - y - z) + 2z - 1 - \lambda = 0$$

$$18y + 3 - 3y - 3z + 2z - 1 - \lambda = 0$$

$$15y - z + 2 - \lambda = 0 \quad (2)$$

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$$(1) - (2)$$

$$15y + 5z - 6 - \lambda - 15y + z - 2 + \lambda = 0$$

$$6z - 8 = 0$$

$$z = \frac{4}{3}$$

$$15y - \frac{4}{3} + 2 - \lambda = 0 \Leftrightarrow \lambda = 15y + \frac{2}{3}$$

$$-7(1 - y - z) + 2y + 2 - 15y - \frac{2}{3} = 0$$

$$-7 + 7y + 7z + 2y + 2 - 15y - \frac{2}{3} = 0$$

$$9y + 7 * \frac{4}{3} - \frac{17}{3} - 15y = 0$$

$$6y = \frac{11}{3} \Leftrightarrow y = \frac{11}{18}$$

$$x = 1 - y - z \Leftrightarrow x = 1 - \frac{4}{3} - \frac{11}{18} = -\frac{17}{18}$$

$$\forall (h, i, j) \in \mathbb{R}^3 \text{ avec } -\frac{17}{18} + h + \frac{11}{18} + i + \frac{4}{3} + j - 1 = 0 \Leftrightarrow h + i + j = 0$$

$$\begin{aligned} F\left(-\frac{17}{18} + h, \frac{11}{18} + i, \frac{4}{3} + j\right) - F\left(-\frac{17}{18}, \frac{11}{18}, \frac{4}{3}\right) \\ = -6\left(-\frac{17}{18} + h\right)^2 + 9\left(\frac{11}{18} + i\right)^2 + 3\left(-\frac{17}{18} + h\right)\left(\frac{11}{18} + i\right) \\ - 7\left(-\frac{17}{18} + h\right)\left(\frac{4}{3} + j\right) + 2\left(\frac{11}{18} + i\right)\left(\frac{4}{3} + j\right) + 6\left(-\frac{17}{18} + h\right) - \left(\frac{11}{18} + i\right) \\ + 2\left(\frac{4}{3} + j\right) - \frac{28}{9} \end{aligned}$$

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$$\begin{aligned}
&= -6 \left( \frac{289}{324} - \frac{17}{9}h + h^2 \right) + 9 \left( \frac{121}{324} + \frac{11}{9}i + i^2 \right) + 3 \left( \frac{-187}{324} - \frac{17}{18}i + \frac{11}{18}h + ih \right) \\
&\quad - 7 \left( \frac{-34}{27} - \frac{17}{18}j + \frac{4}{3}h + hj \right) + 2 \left( \frac{22}{27} + \frac{11}{18}j + \frac{4}{3}i + ij \right) - \frac{17}{3} + 6h - \frac{11}{18} - i \\
&\quad + \frac{8}{3} + 2j - \frac{28}{9} \\
&= \frac{-289}{54} + \frac{34}{3}h - 6h^2 + \frac{121}{36} + 11i + 9i^2 - \frac{187}{108} - \frac{17}{6}i + \frac{11}{6}h + 3ih + \frac{238}{27} + \frac{119}{18}j - \frac{28}{3}h \\
&\quad - 7hj + \frac{44}{27} + \frac{11}{9}j + \frac{8}{3}i + 2ij - \frac{17}{3} + 6h - \frac{11}{18} - i + \frac{8}{3} + 2j - \frac{28}{9} \\
&= \frac{59}{6}h + \frac{59}{6}i + \frac{59}{6}j - 6h^2 + 9i^2 + 3ih - 7hj + 2ij
\end{aligned}$$

Or, d'après la contrainte,  $h + i + j = 0$  donc  $\frac{59}{6}h + \frac{59}{6}i + \frac{59}{6}j = 0$

$$\begin{aligned}
&= -6h^2 + 9i^2 + 3ih - 7hj + 2ij \\
&= -6h^2 + 9i^2 + 3ih - 7h(-i - h) + 2i(-i - h) \\
&= -6h^2 + 9i^2 + 3ih + 7ih + 7h^2 - 2i^2 - 2ih \\
&= h^2 + 7i^2 + 8ih
\end{aligned}$$

Or, le signe de  $h^2 + 7i^2 + 8ih$  dépend du signe de  $i$  et de  $h$ . Donc, on ne peut pas optimiser la fonction sous la contrainte  $x + y + z = 1$ .

Exercice 5

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq x, 0 \leq y, 0 \leq z, x + y + z \leq 1\}$$

$$\text{Donc } 0 \leq z \leq 1, 0 \leq y \leq 1 - z, 0 \leq x \leq 1 - z - y$$

$$\begin{aligned}
I &= \int_{z=0}^{z=1} \left( \int_{y=0}^{y=1-z} \left( \int_{x=0}^{x=1-z-y} -6x^2 + 9y^2 + 3xy - 7xz + 2yz + 6x - y + 2z \, dx \right) dy \right) dz \\
I &= \int_{z=0}^{z=1} \left( \int_{y=0}^{y=1-z} \left[ -2x^3 + 9y^2x + \frac{3}{2}x^2y - \frac{7}{2}x^2z + 2xyz + 3x^2 - yx + 2zx \right]_{x=0}^{x=1-z-y} dy \right) dz \\
&\quad \left[ -2x^3 + 9y^2x + \frac{3}{2}x^2y - \frac{7}{2}x^2z + 2xyz + 3x^2 - yx + 2zx \right]_{x=0}^{x=1-z-y} \\
&\quad = \frac{7}{2}y - \frac{3}{2}z + 1 + \frac{7}{2}z^2y - \frac{3}{2}z^3 + z^2 + \frac{7}{2}y^3 - \frac{3}{2}zy^2 + y^2 + 7zy^2 - 3z^2y \\
&\quad + 2zy - 7zy + 3z^2 - 2z - 7y^2 - 3yz - 2y \\
&= \frac{3}{2}y - \frac{7}{2}z + 1 + 4z^2 - 6y^2 - \frac{3}{2}z^3 + \frac{7}{2}y^3 - 8zy + \frac{1}{2}z^2y + \frac{11}{2}zy^2
\end{aligned}$$

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$$= -\frac{3}{2}z^3 + \frac{7}{2}y^3 + z^2\left(4 + \frac{1}{2}y\right) + y^2\left(-6 + \frac{11}{2}z\right) + y\left(\frac{3}{2} - 8z\right) - \frac{7}{2}z + 1$$

$$I = \int_{z=0}^{z=1} \int_{y=0}^{y=1-z} -\frac{3}{2}z^3 + \frac{7}{2}y^3 + z^2\left(4 + \frac{1}{2}y\right) + y^2\left(-6 + \frac{11}{2}z\right) + y\left(\frac{3}{2} - 8z\right) - \frac{7}{2}z + 1 \, dy \, dz$$

$$I = \int_{z=0}^{z=1} \left[ -\frac{3}{2}yz^3 + \frac{7}{8}y^4 + z^2\left(4y + \frac{1}{4}y^2\right) + \frac{1}{3}y^3\left(-6 + \frac{11}{2}z\right) + \frac{1}{2}y^2\left(\frac{3}{2} - 8z\right) - \frac{7}{2}zy + y \right]_{y=0}^{y=1-z}$$

$$\begin{aligned} & \left[ -\frac{3}{2}yz^3 + \frac{7}{8}y^4 + z^2\left(4y + \frac{1}{4}y^2\right) + \frac{1}{3}y^3\left(-6 + \frac{11}{2}z\right) + \frac{1}{2}y^2\left(\frac{3}{2} - 8z\right) - \frac{7}{2}zy + y \right]_{y=0}^{y=1-z} \\ &= (1-z)\left(-\frac{3}{2}z^3 - \frac{7}{2}z + 1\right) + \frac{7}{8}(1-z)^4 \\ &+ z^2\left(4(1-z) + \frac{1}{4}(1-2z+z^2)\right) + \frac{1}{3}(1-z)^3\left(-6 + \frac{11}{2}z\right) + \frac{1}{2}(1-z)^2\left(\frac{3}{2} - 8z\right) \end{aligned}$$

$$\begin{aligned} &= (1-z)\left(-\frac{3}{2}z^3 - \frac{7}{2}z + 1\right) + \frac{7}{8}(1-4z+6z^2-4z^3+z^4) \\ &+ z^2\left(4-4z+\frac{1}{4}-\frac{1}{2}z+\frac{1}{4}z^2\right) + \frac{1}{3}(1-3z+3z^2-z^3)\left(-6 + \frac{11}{2}z\right) + \frac{1}{2}(1-2z+z^2)\left(\frac{3}{2} - 8z\right) \end{aligned}$$

$$\begin{aligned} &= -\frac{3}{2}z^3 - \frac{7}{2}z + 1 + \frac{3}{2}z^4 + \frac{7}{2}z^2 - z + \frac{7}{8} - \frac{7}{2}z + \frac{21}{4}z^2 - \frac{7}{2}z^3 + \frac{7}{8}z^4 + 4z^2 - 4z^3 + \frac{1}{4}z^2 \\ &- \frac{1}{2}z^3 + \frac{1}{4}z^4 + \frac{1}{3}\left(-6 + \frac{11}{2}z + 18z - \frac{33}{2}z^2 - 18z^2 + \frac{33}{2}z^3 + 6z^3 - \frac{11}{2}z^4\right) \\ &+ \frac{1}{2}\left(\frac{3}{2} - 8z - 3z + 16z^2 + \frac{3}{2}z^2 - 8z^3\right) \end{aligned}$$

$$\begin{aligned} &= -\frac{3}{2}z^3 - \frac{7}{2}z + 1 + \frac{3}{2}z^4 + \frac{7}{2}z^2 - z + \frac{7}{8} - \frac{7}{2}z + \frac{21}{4}z^2 - \frac{7}{2}z^3 + \frac{7}{8}z^4 + 4z^2 - 4z^3 + \frac{1}{4}z^2 \\ &- \frac{1}{2}z^3 + \frac{1}{4}z^4 - 2 + \frac{11}{6}z + 6z - \frac{11}{2}z^2 - 6z^2 + \frac{11}{2}z^3 + 2z^3 - \frac{11}{6}z^4 + \frac{3}{4} \\ &- 4z - \frac{3}{2}z + 8z^2 + \frac{3}{4}z^2 - 4z^3 \end{aligned}$$

$$\begin{aligned} &= z^4\left(\frac{3}{2} + \frac{7}{8} + \frac{1}{4} - \frac{11}{6}\right) + z^3\left(-\frac{3}{2} - \frac{7}{2} - 4 - \frac{1}{2} + \frac{11}{2} + 2 - 4\right) \\ &+ z^2\left(\frac{7}{2} + \frac{21}{4} + 4 + \frac{1}{4} - \frac{11}{2} - 6 + 8 + \frac{3}{4}\right) \\ &+ z\left(-\frac{7}{2} - 1 - \frac{7}{2} + \frac{11}{6} + 6 - 4 - \frac{3}{2}\right) + 1 + \frac{7}{8} - 2 + \frac{3}{4} \end{aligned}$$

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$$= \frac{19}{24}z^4 - 6z^3 + \frac{41}{4}z^2 - \frac{17}{3}z + \frac{5}{8}$$

$$I = \int_{z=0}^{z=1} \frac{19}{24}z^4 - 6z^3 + \frac{41}{4}z^2 - \frac{17}{3}z + \frac{5}{8} dz$$

$$I = \left[ \frac{19}{120}z^5 - \frac{3}{2}z^4 + \frac{41}{12}z^3 - \frac{17}{6}z^2 + \frac{5}{8}z \right]_{z=0}^{z=1}$$

$$I = -\frac{2}{15}$$