

$$1) \int_0^a |\psi(x)|^2 dx = 1 \Leftrightarrow \int_0^a |N x(a-x)|^2 dx = 1$$

$$\Leftrightarrow |N|^2 \int_0^a |x a - x^2|^2 dx = 1$$

$$\Leftrightarrow |N|^2 \int_0^a (x a - x^2)^2 dx = 1 \quad \text{car } x \in [0, a]$$

$$\Leftrightarrow |N|^2 \int_0^a x^2 a^2 - 2x^3 a + x^4 dx = 1$$

$$\Leftrightarrow |N|^2 \left[\frac{1}{3} x^3 a^2 - \frac{1}{2} x^4 a + \frac{1}{5} x^5 \right]_0^a = 1$$

$$\Leftrightarrow |N|^2 \frac{a^5}{30} = 1 \Leftrightarrow |N|^2 = \frac{30}{a^5} \Leftrightarrow \boxed{N = \sqrt{\frac{30}{a^5}}}$$

$$2) \text{ Position moyenne : } \langle x \rangle = \int_0^a x |\psi|^2 dx$$

$$\frac{\partial \psi}{\partial x} = Na - 2Nx$$

$$= |N|^2 \int_0^a x^3 a^2 - 2a x^4 + x^5 dx$$

$$= |N|^2 \left[\frac{1}{4} x^4 a^2 - \frac{2}{5} a x^5 + \frac{1}{6} x^6 \right]_0^a$$

$$= \frac{|N|^2 a^6}{60} \quad \text{avec } N = \sqrt{\frac{30}{a^5}} \Rightarrow \boxed{\langle x \rangle = \frac{1}{2} a}$$

$$\text{Impulsion moyenne : } \langle p \rangle = \int_0^a \psi^* \left(-\frac{\hbar}{i} \right) \frac{\partial \psi}{\partial x} dx$$

9
0

$$= \int_0^a (N x(a-x)) \left(-\frac{\hbar}{i} \right) (-N(x-a) - N x) dx$$

$$= -\frac{\hbar}{i} \int_0^a N^2 x(a-x)(2x-a) dx$$

$$= -N^2 \frac{\hbar}{i} \left[2x^2 a - 2x^3 \right]_0^a = \boxed{0}$$

$$\begin{aligned} & N x(a-x) (N(a-x) - N x) \\ & N^2 x(a-x)^2 - N^2 x^2 \\ & N^2 x(a^2 - 2ax + x^2) - N^2 x^2 \\ & N^2 x a^2 - 2N^2 a x^2 + N^2 x^3 - N^2 x^2 \\ & (N^2 x a - N^2 x^2)(2x-a) \\ & 2N^2 x^2 a - N^2 a^2 x - 2N^2 x^3 \\ & + N^2 x^2 a \end{aligned}$$

3) Incertitude sur la position :

→ calcul de $\langle x^2 \rangle$

$$\langle x^2 \rangle = \int_0^a \psi^*(x) x^2 \psi(x) dx$$

avec $\hat{x} = x$

$$= \int_0^a \psi(x^2) x^2 dx$$

$$= N^2 \int_0^a x^2 (a-x)^2 x^2 dx = N^2 \int_0^a x^4 (a-x)^2 dx$$

$$= N^2 \int_0^a x^4 a^2 - 2a x^5 + x^6 dx$$

$$= N^2 \left[\frac{1}{5} a^2 x^5 - \frac{1}{3} a x^6 + \frac{1}{7} x^7 \right]_0^a = \frac{30 a^7}{a^5 \cdot 105} = \boxed{\frac{2a^2}{7}}$$

$$\text{Incertitude sur la position : } \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{2a^2}{7} - \frac{1}{4} a^2} = \boxed{\sqrt{\frac{1}{28} a^2}}$$

Incertitude sur l'impulsion: $\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$

→ Calcul de $\langle p^2 \rangle$

$$\langle p^2 \rangle = \int_0^a \psi^*(x,t) \hat{p}^2 \psi(x,t) dx$$

$$= \int_0^a N x(x-a) \hat{p}^2 \psi(x,t) dx$$

$$= \int_0^a N x(x-a) \hbar^2 \cdot 2N dx$$

$$= \hbar^2 2N^2 \int_0^a x(a-x) dx$$

$$= \hbar^2 2N^2 \left[\frac{1}{2} x^2 a - \frac{1}{3} x^3 \right]_0^a$$

$$= \hbar^2 2N^2 \frac{1}{6} a^3 = \hbar^2 2 \cdot \frac{30}{a^5} \cdot \frac{1}{6} a^3 = \boxed{\hbar^2 \cdot 10 \cdot \frac{1}{a^2}}$$

$$\Delta p = \sqrt{\hbar^2 \cdot \frac{10}{a^2} - 0} = \boxed{\frac{\hbar}{a} \sqrt{10}}$$

$$\hat{p}^2 \psi(x,t) = \left(-i\hbar \frac{\partial}{\partial x} \right)^2 \psi(x,t) \\ = -\hbar^2 \frac{\partial^2}{\partial x^2} \psi(x,t)$$

$$\boxed{\hat{p}^2 \psi(x,t) = -\hbar^2 \cdot 2N}$$