TAI 5 particule confinée entre $0 \le x \le a$.

$$\psi(x) = Nx(a-x)$$

1) $\int_0^a |\psi(x)|^2 dx = 1 \Rightarrow \int_0^a [Nxa - Nx^2]^2 dx = 1$

$$\Rightarrow \int_0^a (Nxa)^2 dx + \int_0^a (Nx^2)^2 dx - \int_0^a 2N^2 a x^3 dx = 1$$

$$\Rightarrow N^2a^2\left[\frac{x^3}{3}\right]_0^a + N^2\left[\frac{x^5}{5}\right]_0^a - 2N^2a\left[\frac{x^4}{4}\right]_0^a = 1$$

$$\Rightarrow N^2\left(\frac{a^5}{3} + \frac{a^5}{5} - \frac{a^5}{2}\right) = 1$$

$$\Rightarrow N^2\left(\frac{a^5}{30}\right) = 1$$

$$\Rightarrow N = \sqrt{\frac{30}{a^5}}$$

4) Position moyenne : $\langle x \rangle = \int_0^a x |\psi(x)|^2 dx$

$$\langle x \rangle = \int_0^a x |Nx(a-x)|^2 dx = |N|^2 \int_a^0 x^3 (a^2 - 2ax + x^2) dx$$

$$\langle x \rangle = |N|^2 \int_0^a (x^5 - 2ax^4 + x^3a^2) dx$$

$$\langle x \rangle = |N|^2 \left[\frac{x^6}{6} - \frac{2ax^5}{5} + \frac{a^2x^4}{4}\right]_0^a = |N|^2 \left[\frac{a^6}{6} - \frac{2a^6}{5} + \frac{a^6}{4}\right]$$

$$\langle x \rangle = |N|^2 \left[\frac{10a^6 - 24a^6 + 15a^6}{60}\right] = |N|^2 \frac{a^6}{60} = \sqrt{\frac{30}{a^5}}^2 \times \frac{a^6}{60} = \frac{30}{a^5} \times \frac{a^6}{60} = \frac{1}{2}a$$

Impulsul mediu:

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^*(x) \left| \frac{-\hbar}{i} \right| \frac{d\psi}{dx} dx \qquad \frac{\partial \psi}{\partial x} = (Na - 2Nx)$$

$$\langle p \rangle = \int_0^a Nx(a-x) \left| \frac{\hbar}{i} \right| (Na - 2Nx)\, dx$$

$$\langle p \rangle = \int_0^a N^2 x(a-x) \left| \frac{\hbar}{i} \right| (a - 2x)\, dx$$

$$\langle p \rangle = \int_0^a N^2 (xa - x^2)(a - 2x) \left| \frac{\hbar}{i} \right| dx$$

$$\langle p \rangle = N^2 \left| \frac{\hbar}{i} \right| \int_0^a xa^2 - 2ax^2 - x^2 a + 2x^3\, dx$$

$$\langle p \rangle = N^2 \left| \frac{\hbar}{i} \right| \int_0^a 2x^3 - 3ax^2 + xa^2\, dx$$

$$\langle p \rangle = N^2 \left| \frac{\hbar}{i} \right| \left[\frac{x^4}{2} - ax^3 + \frac{a^2 x^2}{2} \right]_0^a = N^2 \left| \frac{\hbar}{i} \right| \left(\frac{a^4}{2} - \frac{2a^4}{2} + \frac{a^4}{2} \right)$$

$$\langle p \rangle = N^2 \left| \frac{\hbar}{i} \right| \times 0$$

$$\boxed{\langle p \rangle = 0}$$

3) Incertitude de position

$$\langle x^2 \rangle = \int_0^a \psi^*(x)\, \hat{x}^2 \psi(x)\, dx \qquad \text{avec } \hat{x}^2 = x^2 \quad \psi^* = \psi$$

$$\langle x^2 \rangle = \int_0^a x^2 \psi(x)^2\, dx \qquad \psi^2 = (Nx(a-x))^2 = N^2 x^2 (a^2 - 2ax + x^2)$$

$$\langle x^2 \rangle = \int_0^a x^2 \left(N^2 x^2 (a^2 - 2ax + x^2)\right) dx$$

$$\langle x^2 \rangle = \int_0^a N^2 x^4 (a^2 - 2ax + x^2)\, dx = N^2 \int_0^a x^4 a^2 - 2a x^5 + x^6\, dx$$

$$\langle x^2 \rangle = N^2 \left[\frac{a^2 x^5}{5} - \frac{a x^6}{3} + \frac{x^7}{7}\right]_0^a = N^2 \left(\frac{a^7}{5} - \frac{a^7}{3} + \frac{a^7}{7}\right) = N^2 \left(\frac{21a^7 - 35a^7 + 15a^7}{105}\right)$$

$$\langle x^2 \rangle = \sqrt{\frac{30}{a^5}}^2 \left(\frac{a^7}{105}\right) = \frac{30\, a^2}{105} = \boxed{\frac{2a^2}{7}}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{2a^2}{7} - \left(\frac{1}{2}a\right)^2} = \sqrt{\frac{2a^2}{7} - \frac{1}{4}a^2}$$

$$\Delta x = \sqrt{\frac{8a^2 - 7a^2}{28}} = \sqrt{\frac{a^2}{28}} = \boxed{\frac{a}{2\sqrt{7}}}$$

$\Delta x \Delta p \geq \frac{\hbar}{2}$ ⇒ cinématique minimum : $\Delta p = \frac{\hbar}{2\Delta x}$

$$\Delta p = \frac{\hbar}{2} \times \frac{2\sqrt{7}}{a} = \boxed{\frac{\hbar\sqrt{7}}{a}}$$