

ex 6

$\langle p^2 \rangle \Delta x$

1) Proba de trouver la particule ds l'espace dx : $dP(x) = |\Phi(x)|^2 dx$
pour $x \geq 0$

Proba de trouver entre x et $x+dx$: $P(x) = \int_x^{x+dx} dP(x) dx$

$$|\Phi(x)|^2 = \Phi^*(x) \cdot \Phi(x)$$

$$= N^2 x^2 e^{-\alpha x}$$

$$= \frac{\alpha^3}{2} x^2 e^{-\alpha x}$$

$$\Rightarrow P(x) = \frac{\alpha^3}{2} \int_x^{x+dx} x^2 e^{-\alpha x} dx$$

$$\langle x \rangle = \int_0^{+\infty} x \cdot |\Phi(x)|^2 dx = \frac{\alpha^3}{2} \int_0^{+\infty} x^3 e^{-\alpha x} dx = \frac{\alpha^3}{2} \frac{3!}{\alpha^4} = \boxed{\frac{3}{\alpha}}$$

$$\langle x^2 \rangle = \int_0^{+\infty} x^2 |\Phi(x)|^2 dx = \frac{\alpha^3}{2} \int_0^{+\infty} x^4 e^{-\alpha x} dx = \frac{\alpha^3}{2} \frac{4!}{\alpha^5} = \boxed{\frac{12}{\alpha^2}}$$

$$2) \langle p \rangle = \int_0^{+\infty} \Phi(x) \frac{\hbar}{i} \frac{d\Phi(x)}{dx} dx$$

$$\int x^n e^{-\alpha x} dx = \frac{n!}{\alpha^{n+1}}$$

$$\text{On } \frac{d\Phi(x)}{dx} = N(e^{-\alpha x/2} + x(-\frac{\alpha}{2})e^{-\alpha x/2})$$

$$\text{D'où } \langle p \rangle = -i\hbar \int_0^{+\infty} N x e^{-\alpha x/2} \cdot N(e^{-\alpha x/2} + x(-\frac{\alpha}{2})e^{-\alpha x/2}) dx$$

$$\Rightarrow \langle p \rangle = -i\hbar N^2 \left(\int_0^{+\infty} x e^{-\alpha x} dx - \frac{\alpha}{2} \int_0^{+\infty} x^2 e^{-\alpha x} dx \right)$$

$$\Rightarrow \langle p \rangle = -i\hbar N^2 \left(\frac{1}{\alpha^2} - \frac{\alpha}{2} \cdot \frac{2!}{\alpha^3} \right)$$

$$\Rightarrow \boxed{\langle p \rangle = 0}$$

$$\langle p^2 \rangle = \int_0^{+\infty} \Phi^*(x) \cdot \left(-i\hbar \frac{d}{dx}\right)^2 \Phi(x) dx = -\hbar^2 \int_0^{+\infty} \Phi^*(x) \frac{d^2 \Phi(x)}{dx^2} dx$$

$$\frac{d^2 \Phi(x)}{dx^2} = N \left(-\frac{\alpha}{2} e^{-\alpha x/2} + (-\frac{\alpha}{2}) e^{-\alpha x/2} + \frac{\alpha^2}{4} x e^{-\alpha x/2} \right)$$

$$= N \left(-\frac{\alpha}{2} \right) \left(2e^{-\alpha x/2} - \frac{\alpha}{2} x e^{-\alpha x/2} \right)$$

$$\text{D'où } \langle p^2 \rangle = -\hbar^2 \int_0^{+\infty} N x e^{-\alpha x/2} \cdot N \left(-\frac{\alpha}{2}\right) \left(2e^{-\alpha x/2} - \frac{\alpha}{2} x e^{-\alpha x/2}\right) dx$$

$$\Rightarrow \langle p^2 \rangle = -\hbar^2 N^2 \left(-\frac{\alpha}{2}\right) \left[2 \int_0^{+\infty} x e^{-\alpha x/2} dx - \frac{\alpha}{2} \int_0^{+\infty} x^2 e^{-\alpha x/2} dx \right]$$

$$\Rightarrow \langle p^2 \rangle = \hbar^2 N^2 \frac{\alpha}{2} \left(2 \times \frac{1}{\alpha^2} - \frac{\alpha}{2} \cdot \frac{2}{\alpha^3} \right)$$

$$\Rightarrow \langle p^2 \rangle = \hbar^2 N^2 \left(\frac{1}{\alpha} - \frac{1}{2\alpha} \right)$$

$$\Rightarrow \langle p^2 \rangle = \hbar^2 \frac{\alpha^3}{2} \times \frac{1}{2\alpha}$$

$$\Rightarrow \boxed{\langle p^2 \rangle = \frac{\hbar^2 \alpha^2}{4}}$$

$$3) \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\Rightarrow \Delta x = \sqrt{\frac{12}{\alpha^2} - \frac{9}{\alpha^2}}$$

$$\Rightarrow \Delta x = \frac{\sqrt{3}}{\alpha}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$\Rightarrow \Delta p = \sqrt{\frac{\hbar^2 \alpha^2}{4} - 0}$$

$$\Rightarrow \Delta p = \frac{\hbar \alpha}{2}$$

Principe d'incertitude $\rightarrow$

$$\Delta x \cdot \Delta p = \frac{\sqrt{3}}{\alpha} \cdot \frac{\hbar \alpha}{2} = \boxed{\frac{\hbar}{2} \sqrt{3} \geq \frac{\hbar}{2}}$$