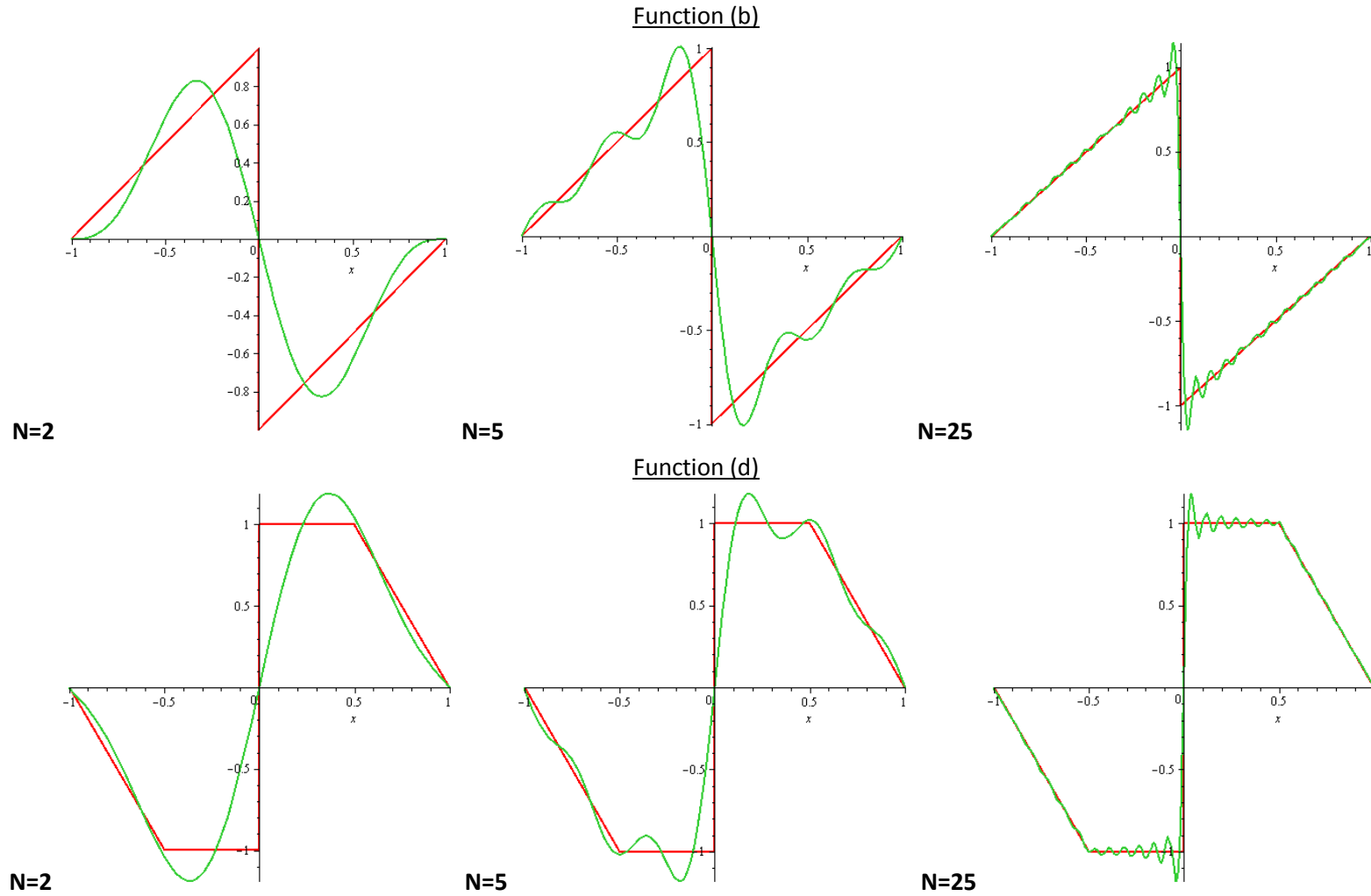


2. Convergence of the series

(a) Comparing and contrasting convergences...

(i) Jumps (Discontinuous)



$$\begin{array}{ll}
 \text{(b)} & f(x) = \begin{cases} x+1 & -1 < x \leq 0 \\ x-1 & 0 < x \leq 1 \end{cases} \\
 \text{(d)} & f(x) = \begin{cases} -2(x+1) & -1 < x \leq -1/2 \\ -1 & -1/2 < x \leq 0 \\ 1 & 0 < x \leq 1/2 \\ -2(x-1) & 1/2 < x \leq 1 \end{cases}
 \end{array}$$

We can see from the graphs that these functions have both **Jumps**.

First observe the function (b):

The jump of the function is due to the transition at $x = 0$ from $(x + 1)$ to $(x - 1)$.

Intervals $x = [-1, 0]$ and $x = [0, 1]$ respectively give values of $f(x) = (x + 1)$ and $f(x) = (x - 1)$. These functions are linear and strictly increasing. Thus, the function takes the form of two straight. The curve corresponding to the Fourier series converges quite rapidly depending on the value of N (the higher N is, the greater the Fourier curve approximates the one of the original function). Indeed, the jump located at $x = 0$ is big (from $f(x) = 1$ to $f(x) = -1$) thereby the Fourier curve for $N = 2$ gives only approximate values : sampling (the number of values calculated the sum) isn't high enough.

At $N = 5$, there is a slight approximation of the Fourier's curve on the linear part of our function, but still unrepresentative around $x = 0$: the jump. For $N = 25$, series of these 25 terms we calculated provide a highly satisfactory representation of our original function! If one small problem should be noted, it would be these small spikes present at both ends of the jump, but we'll talk later (question 2.b).

In brief, the convergence of the series is relatively fast, even slow: after a few test values, a real improvement has been noticed from $N = 14$.

Now let's observe the function (d):

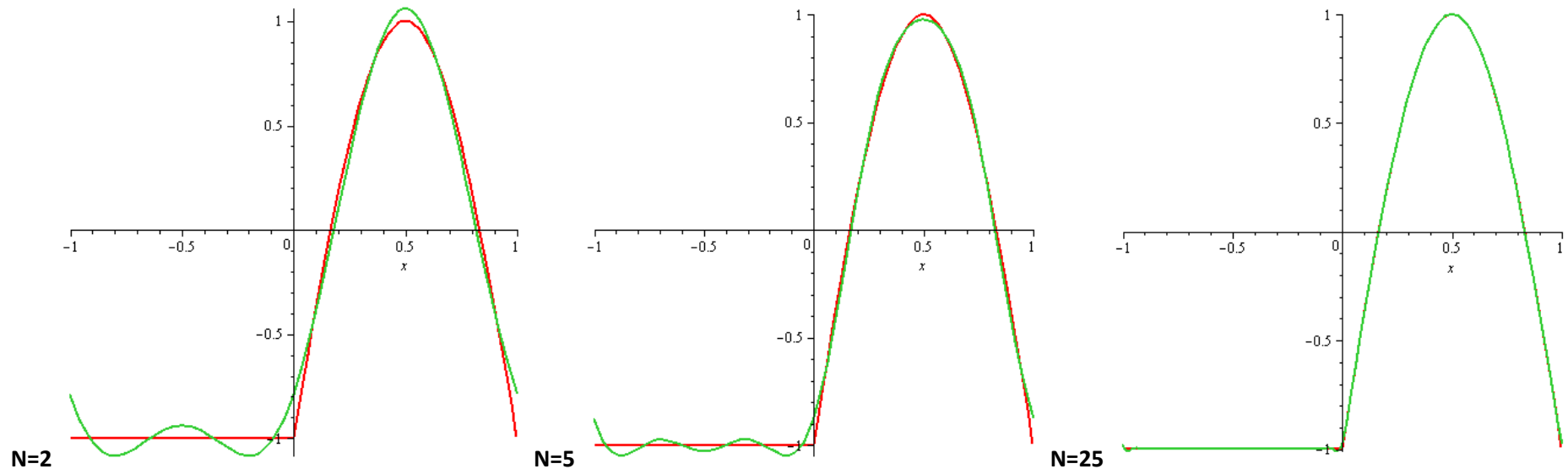
The jump of the function (d) is due to the crossing at $x = 0$ from $f(x) = -1$ to $f(x) = 1$.

In the same way as the function (b), we observe at the two ends two linear functions : a strictly decreasing x in the negative extreme, then a strictly increasing in the positive ($-2(x \pm 1)$). Between these two linear functions, $f(x)$ takes two successive constant values: -1 and $+1$ (hence the jump).

Always similarly to the function (b) the sum of two terms ($N = 2$) is not sufficient, $N = 5$ either; it must actually be between 20 and 22 terms before approaching our function properly! This function is the one with the slowest convergence of all our functions investigated: after several tests, the curves are not only confused about $N = 90$!

(ii) No Jumps (Continuous) , but Corners

Function (a)



$$(a) \quad f(x) = \begin{cases} -1 & -1 < x \leq 0 \\ -1 + 2\sin(\pi x) & 0 < x \leq 1 \end{cases}$$

We can see from the graphs that this function has **No Jump but Corners**.

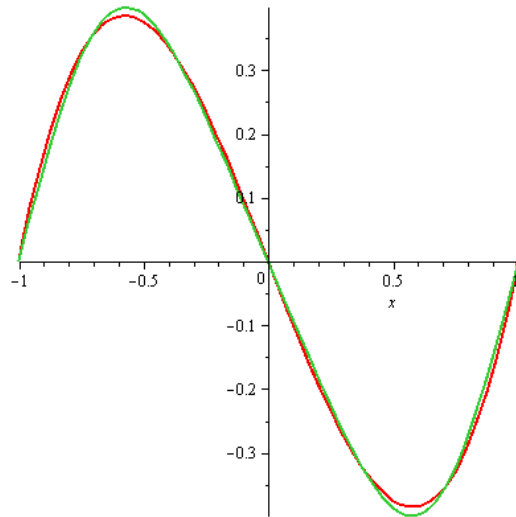
Let's observe the function (a):

There is no jump in this function. Indeed, its first part is a constant: $f(x) = -1$, which extends over a continuous sine function: $-1 + 2\sin(\pi x)$.

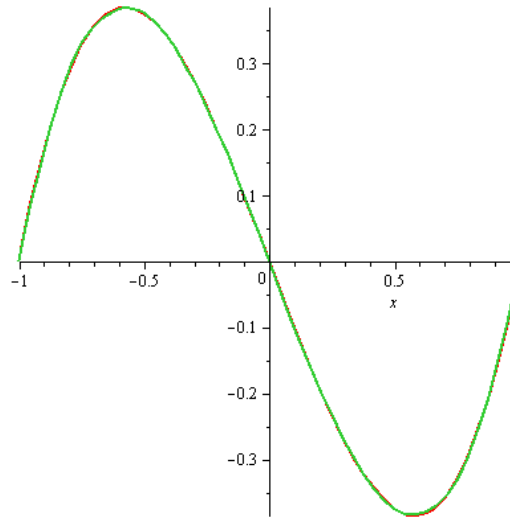
We notice that, in the same way as the linear parts of the previous functions, the Fourier Series is merging easier with our original function, but it takes nonetheless a dozen terms before getting close to our function. Part of the sine curve reveals an entirely new phenomenon: the sine and cosine functions are converging very fast! This is not surprising, because hitch terms of the harmonics (N) given by the formula of Fourier Series is composed by sine and cosine! Indeed, even before $N = 5$ terms, the two curves coincide almost! The only finishing touch is the junction of these two parts of the curve, but less than ten terms is quite enough to confuse the two curves. Needless to say that the two curves are completely identical at the 25th terms.

(iii) No Jumps (Continuous) , and No Corners

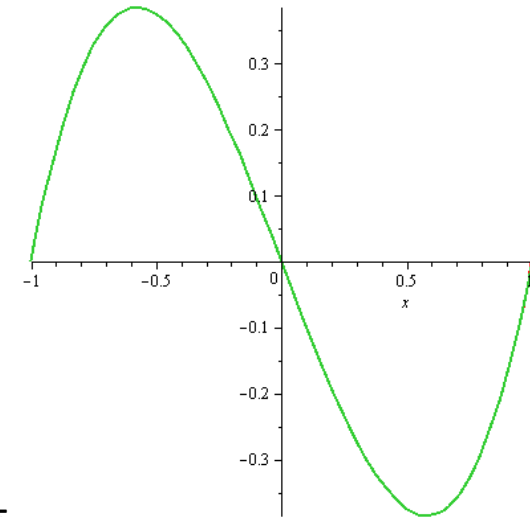
Function (c)



N=2



N=5



N=25

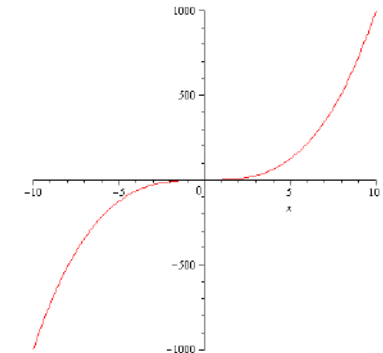
(c) $f(x) = x(x^2 - 1) \quad -1 < x \leq 1$

We can see from the graphs that this function has **No Jump and No Corner**.

Let's observe the function (a):

There is neither jump nor corner in this function. Indeed, this function is made on a continuous interval, the size of the period: $f(x) = x(x^2 - 1)$. The presence of the term x^3 justifies the observation of a bent curve (remember the shape of the function $f(x) = x^3$ the figure opposite).

As observed previously with the sine and cosine functions, the approximation between the curve of Fourier Series and the function we try to recreate is almost immediate: barely 5 terms makes the two curves mixed up! We can thus confidently say that if the function has neither jump nor corner, and it is a bent function such as cosine, sine, or x^α , then the convergence of the Fourier series associated will be very fast!

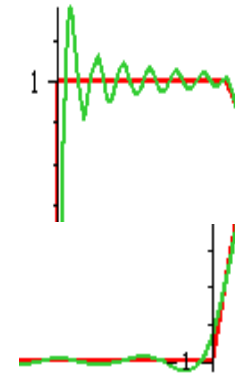


(b) Discussing the convergence for the function(s) with a jump without and including a corner...

When we observe the functions with jumps, we see oscillations appear in the partial sum around points of discontinuity. This phenomenon is called "**Gibbs phenomenon**". It is, somehow, a "defect approximation" of our supposed continuous functions of class C1 (once differentiable) in pieces: the edge effect. It is a phenomenon of numerical instability in the edge of our interval of our parts attempting to calculate an approximate solution to these hotspots.

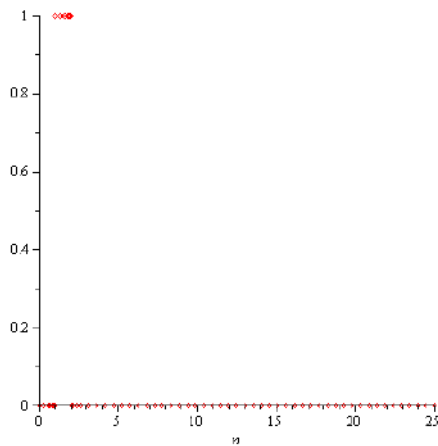
Thus, complete convergence at these points is very slow or impossible.

Whether a jump or a corner, the phenomenon will occur. The convergence of the significant point is faster when it's a corner (see image opposite)

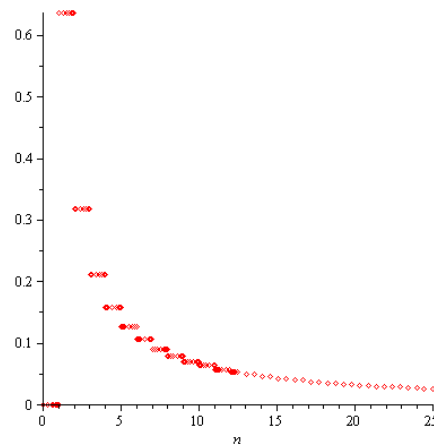


Rate of convergence

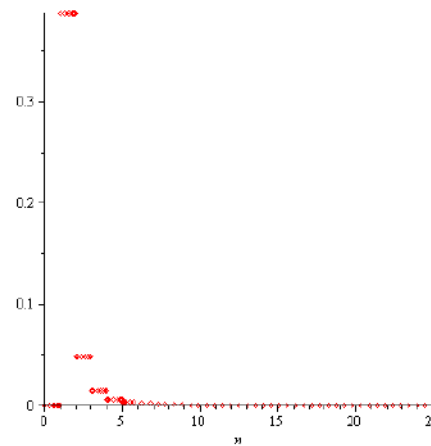
Here are the rates of hitch convergence (a, b, c and d): the magnitude of b_n represented in ordered and N (harmonics) on the abscissa.



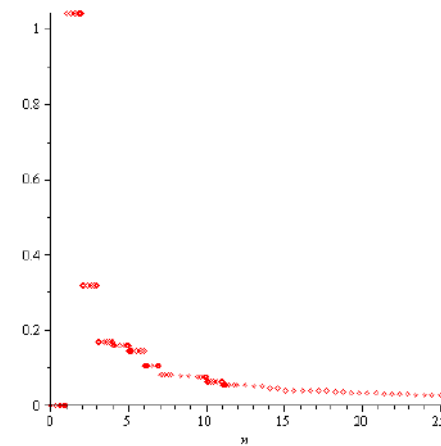
Size of terms decreases suddenly
Terms for large N are irrelevant
Convergence rate is very fast



Size of terms decreases very slowly
Terms for large N are still important
Convergence rate is very slow



Size of terms decreases quickly
Terms for N are quickly unimportant
Convergence rate is relatively fast



Size of terms decreases quietly slowly
Terms for large N are still important
Convergence rate is slow.

(c) Connection between the formula for the series and its convergence...

Here are the coefficients a_0 , a_n and b_n of each function studied. It should be noted that the Fourier Series is governed mainly by the two coefficients a_n and b_n :

$$f(x) = \frac{1}{2}a_0 + \sum_{n=0}^{\infty} \left(a_n \cos\left(\frac{2n\pi x}{T}\right) + b_n \sin\left(\frac{2n\pi x}{T}\right) \right)$$

f_a

$$\begin{aligned} a_0 &= -\frac{2(-2 + \pi)}{\pi} \\ a_n &= \frac{-2((-1)^n - 2)}{\pi n^2 - \pi} \\ b_n &= 0 \end{aligned}$$

f_b

$$\begin{aligned} a_0 &= 0 \\ a_n &= 0 \\ b_n &= -\frac{2}{n\pi} \end{aligned}$$

f_c

$$\begin{aligned} a_0 &= 0 \\ a_n &= 0 \\ b_n &= \frac{12(-1)^n}{n^3\pi^3} \end{aligned}$$

f_d

$$\begin{aligned} a_0 &= 0 \\ a_n &= 0 \\ b_n &= \frac{4 \sin\left(\frac{1}{2} n\pi\right) + 2n\pi}{n^2\pi^2} \end{aligned}$$

For f_a and f_c : We know that their convergences are fast, and this fact is explained by the presence of two key-terms of the Fourier Series' equation: a_n for f_a and b_n for f_b :

- ✓ the numerator in $\lambda \cdot (-1)^n$, causes frequently alternation between positive and negative coefficient.
- ✓ the denominator in $\lambda \cdot n^x$, which causes a significant refinement of the oscillation the larger n is.

For f_b and f_d : We know that their convergence is slow, and this fact is explained by the presence of a single term in the equation of b_n : the $\lambda \cdot n$ denominator. Indeed, f_d can be reduced to $\frac{n}{n^2} = \frac{1}{n}$ and similarly for the function f_c . So the coefficient b_n is making the sum refined by dividing it by a constant, with very slow growth (no exponent).