



Coding and Transformations

Fourier Series

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Introduction

The purpose of this assignment is to determine the distribution of temperature in a solid bar in function of time. Then comment on the shapes of the curves and the evolution in time for different distributions.

This is the three distributions we have to comment:

$$\Phi_1^0 = 4x(1 - x)$$

$$\Phi_2^0 = \begin{cases} 2x, & 0 < x < 1/2 \\ 2 - 2x, & 1/2 \leq x < 1 \end{cases}$$

$$\Phi_3^0 = u(x) - u(x - 1/2)$$

At first we need to calculate the coefficients b_n , then the Fourier Series and plot it to determine a value of N which gives us a good precision.

All curves and graphs are available in the appendices in a bigger format.

Work on Maple

The first step is to find the coefficients b_n . For that, we use Fourier Series and the Heaviside function. The Heaviside function, is defined by:

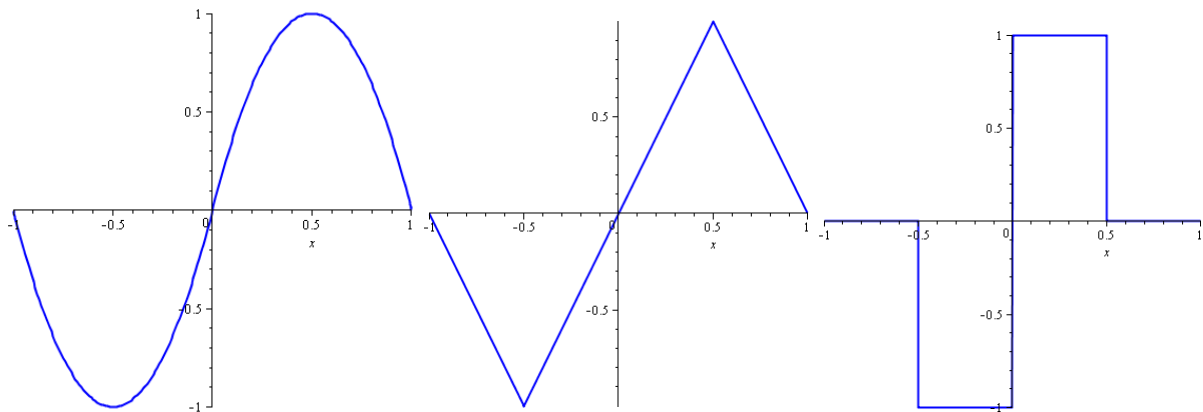
$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

To use Fourier Series, we need a periodic function. This is why we extend the function to obtain an odd periodic function which allows calculating the coefficients b_n . f_1 , f_2 and f_3 are odd periodic functions defined thanks to Heaviside function and corresponding respectively to Φ_1^0 , Φ_2^0 and Φ_3^0 . f_1 and f_2 have a period of 2 and f_3 a period of 1.

$$f_1(x) = -4x(-1-x)(u(x+1) - u(x)) + 4x(1-x)(u(x) - u(x-1))$$

$$f_2(x) = (-2-2x)(u(x+1) - u(x+0.5)) + 2x(u(x+0.5) - u(x-0.5)) \\ + (2-2x)(u(x-0.5) - u(x-1))$$

$$f_3(x) = 2u(x) - u(x-0.5) - u(x+0.5)$$



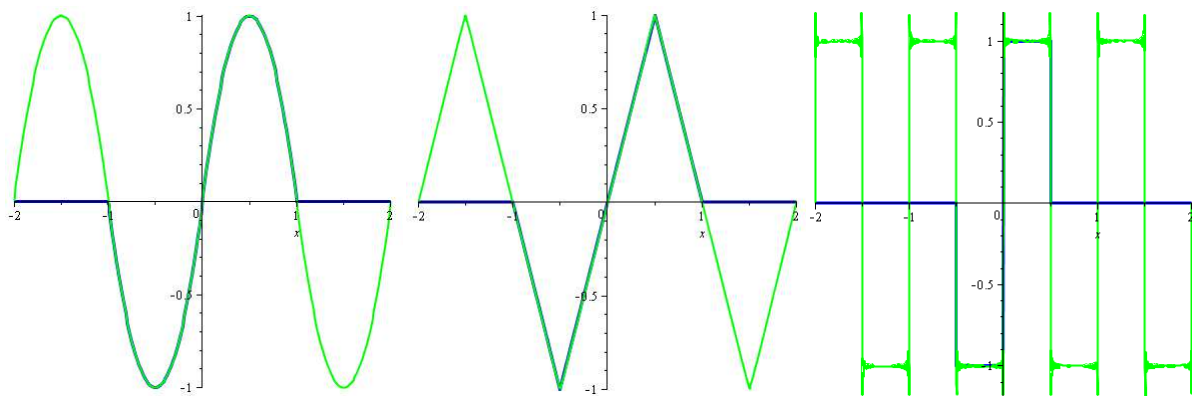
Thanks to these functions, we are now able to calculate the coefficients b_n :

$$b_n = \int_{-1}^1 f(x) \sin(n\pi x) dx$$

With the coefficients b_n , we can find the Fourier Series:

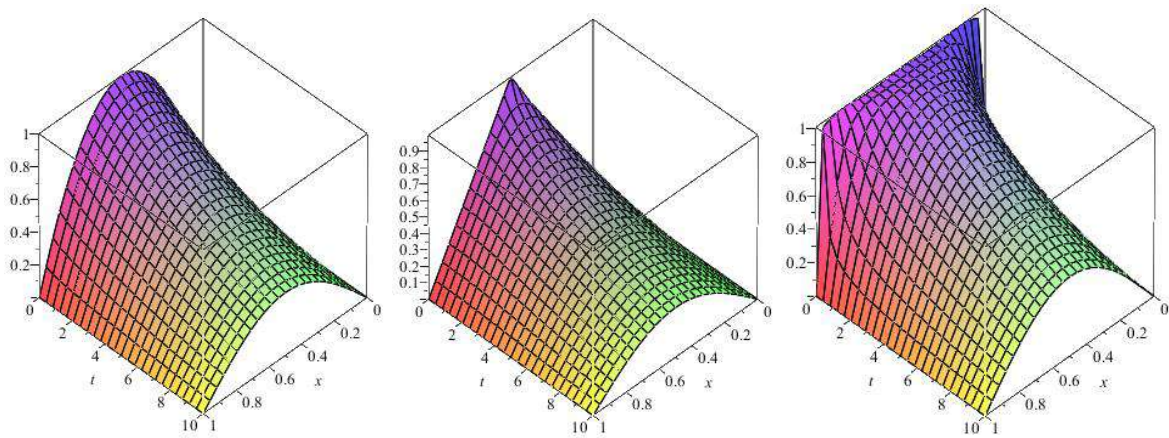
$$F(x) = \sum_{n=1}^N b_n \sin\left(\frac{n\pi x}{T}\right)$$

Here are the three Fourier Series. To have a correct precision, $N = 10$ for f_1 , $N = 100$ for f_2 and $N = 150$ for f_3 . We have to increase N when the function has non-derivable point (function 2) or gap (function 3).



Comments on the results

Thanks to the coefficients b_n we found with each function, we can plot a three-dimensional graph for each distribution. It allows seeing the distribution of the temperature in the solid bar in function of time.



For each function, the temperature on edges is constant and equal to zero. We can also see that the temperature is maximal on the middle of the solid bar at any time.

At the beginning the temperature is way higher on the middle than on edges. Then it seems that temperature tend to be the same on every part of the solid bar.

When $t = 0$, the distribution of temperature of function 1 is parabolic, that of function 2 is more linear and that of function 3 is mostly constant at any point of the solid bar.

When $t = 10$, the temperature on the middle of the solid bar is higher with distribution 3 and lower with distribution 2.

For distribution 1, the temperature decrease linearly. For distribution 2 it decreases faster at the beginning then, when $t = 4$, it decreases slower. For distribution 3 it is almost constant between $t = 0$ and $t = 2$, then it decrease regularly like distribution 1.

Conclusion

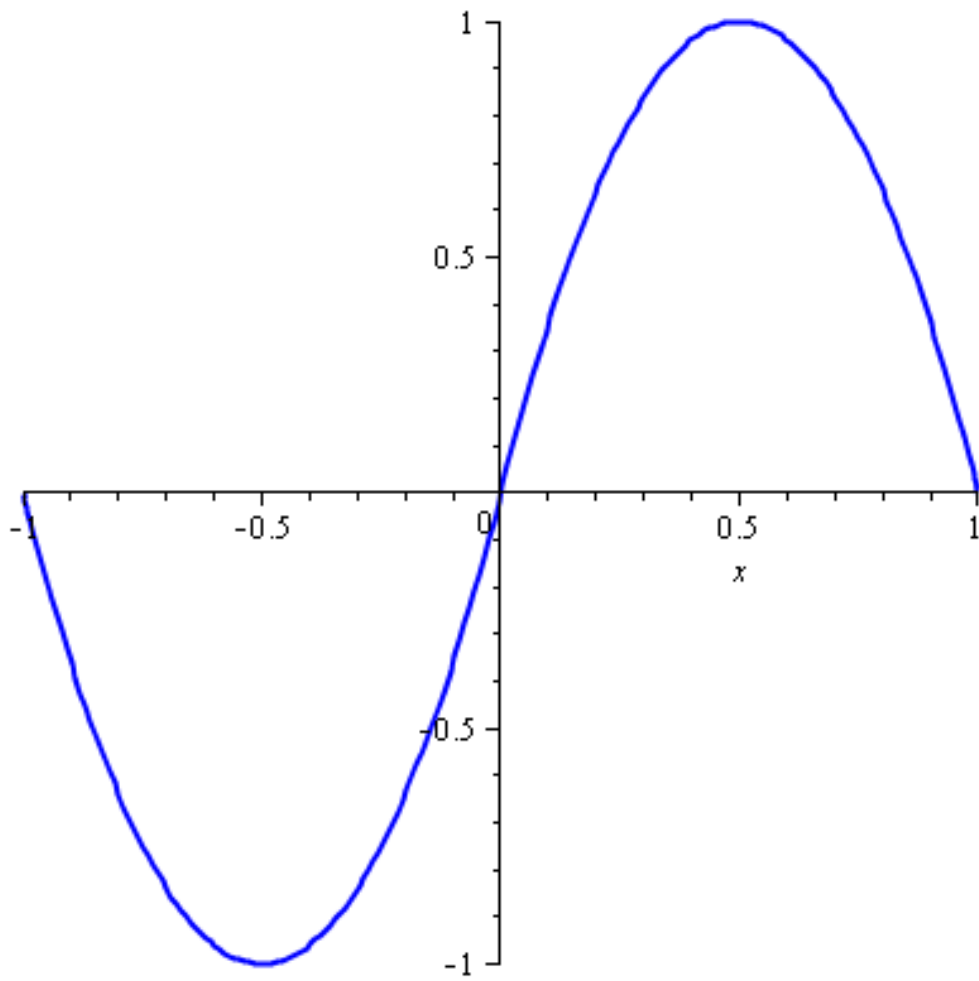
To synthesis on the results, these three distributions have approximately the same comportment. The temperature is always higher in the middle of the solid bar and decrease with time tending to a constant temperature on every point of the solid bar.

These results seem logic and are in agreement with some observable phenomena in our everyday life. When we drink a hot liquid like soup, it is always hottest on the middle than on edges of the plate. We can also take the example of the frozen lake, there always more ice on the lakeside. In this way, closer we get to the center of the lake; greater is the risk to break the ice.

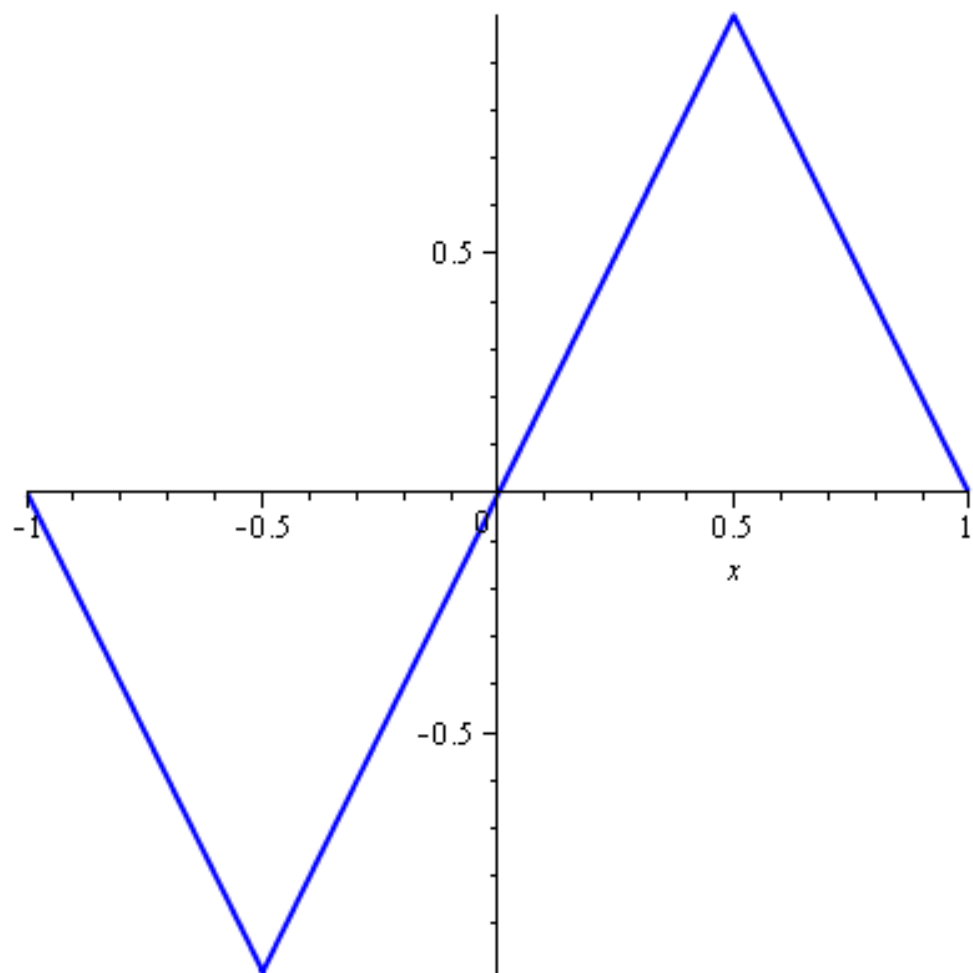
Appendices

Appendices 1: curves of distribution's function

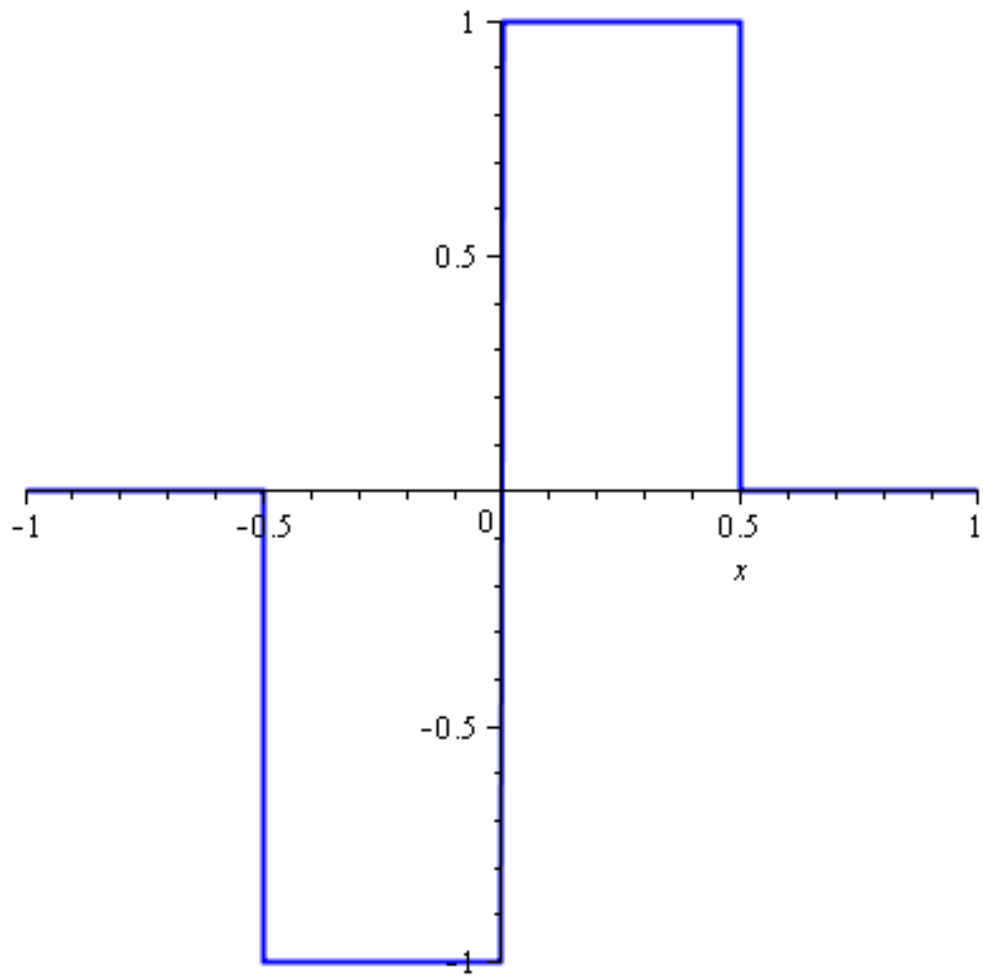
$$f_1(x) = -4x(-1-x)(u(x+1) - u(x)) + 4x(1-x)(u(x) - u(x-1))$$



$$f_2(x) = (-2 - 2x)(u(x + 1) - u(x + 0.5)) + 2x(u(x + 0.5) - u(x - 0.5)) \\ + (2 - 2x)(u(x - 0.5) - u(x - 1))$$

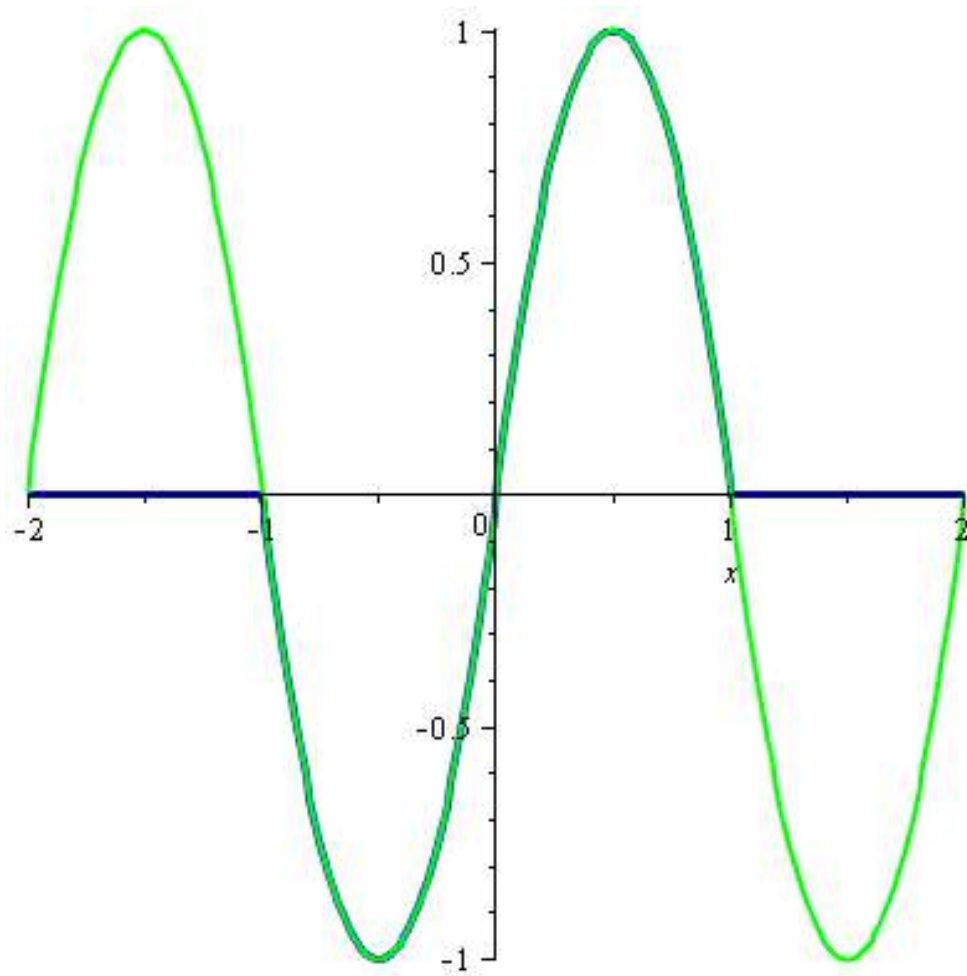


$$f_3(x) = 2u(x) - u(x - 0.5) - u(x + 0.5)$$

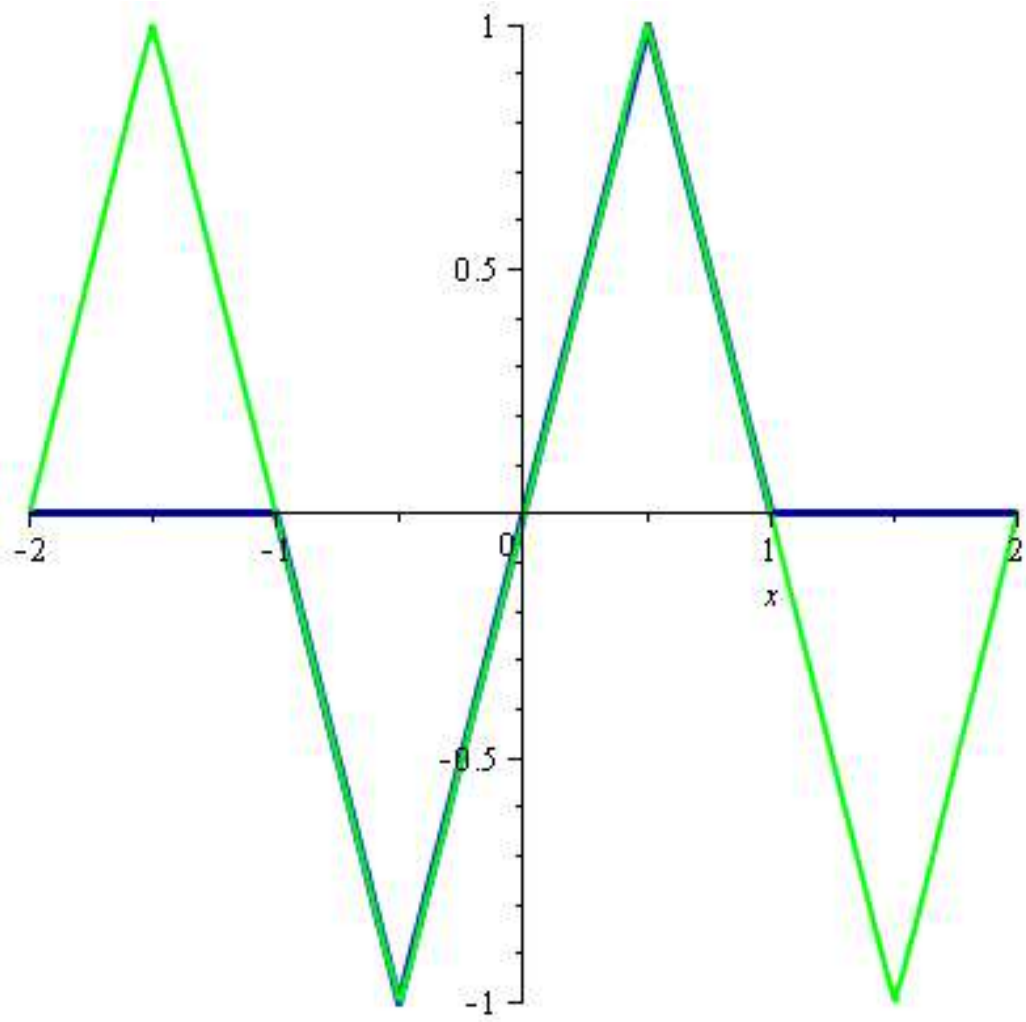


Appendices 2: curves of Fourier Series

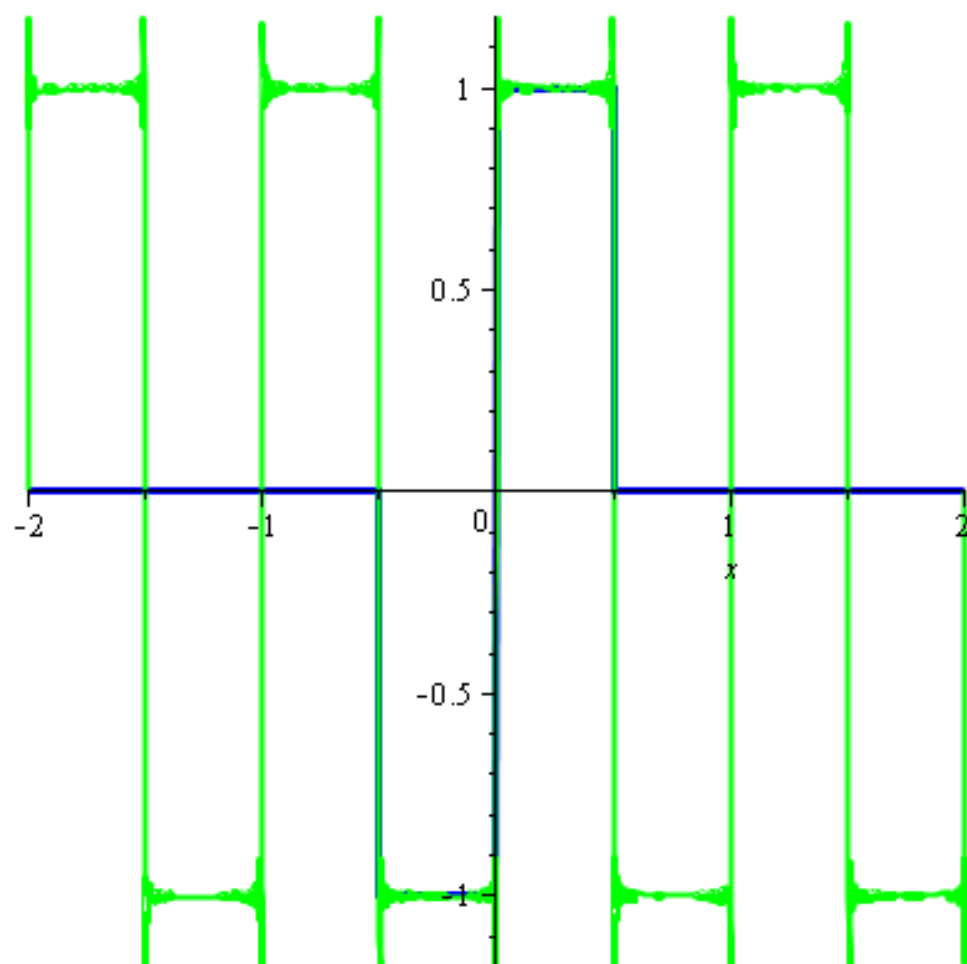
$$F_1(x)$$



$$F_2(x)$$

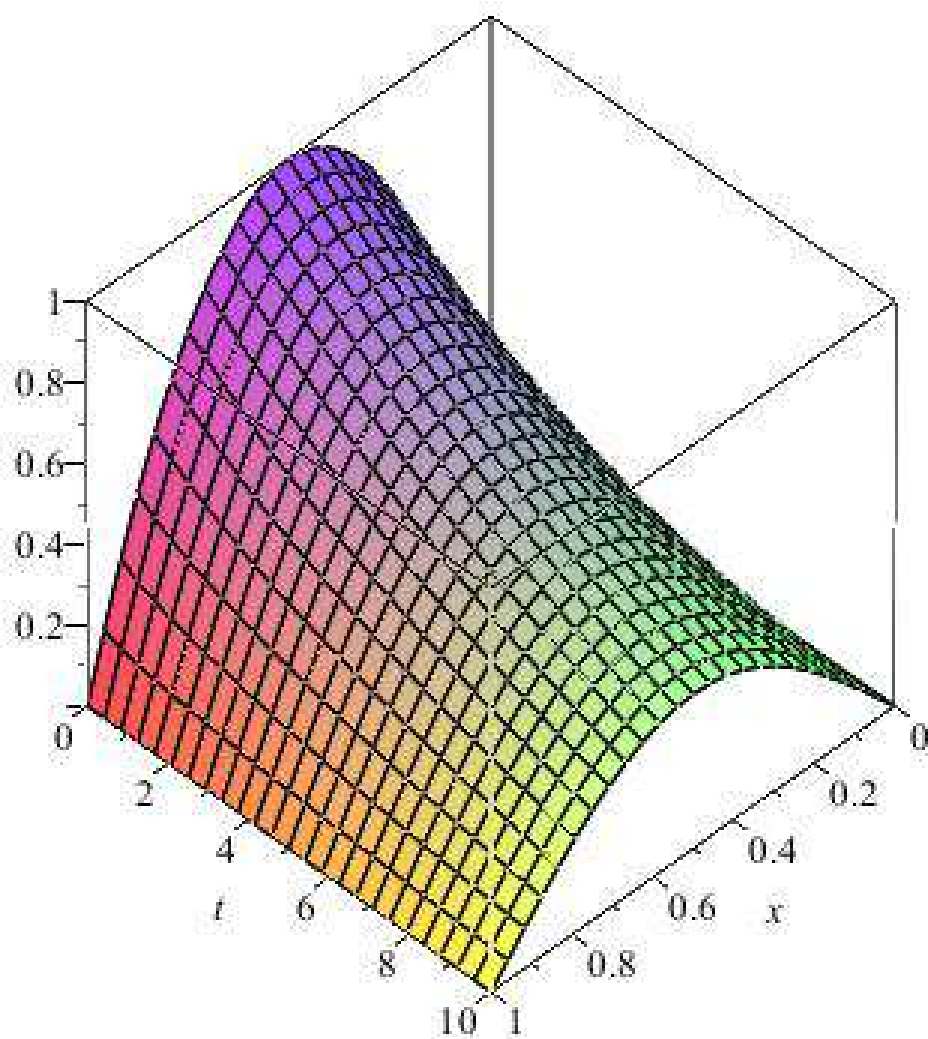


$$F_3(x)$$

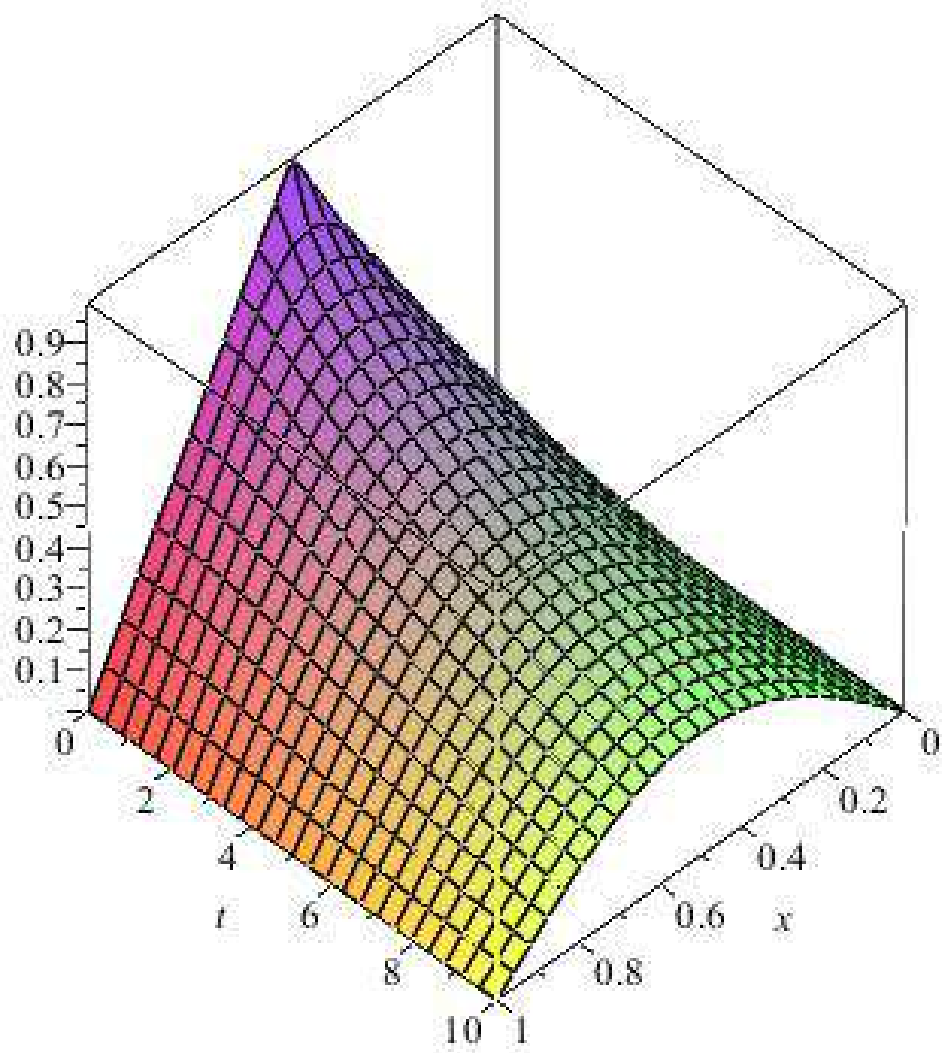


Appendices 3: three-dimensional graphs

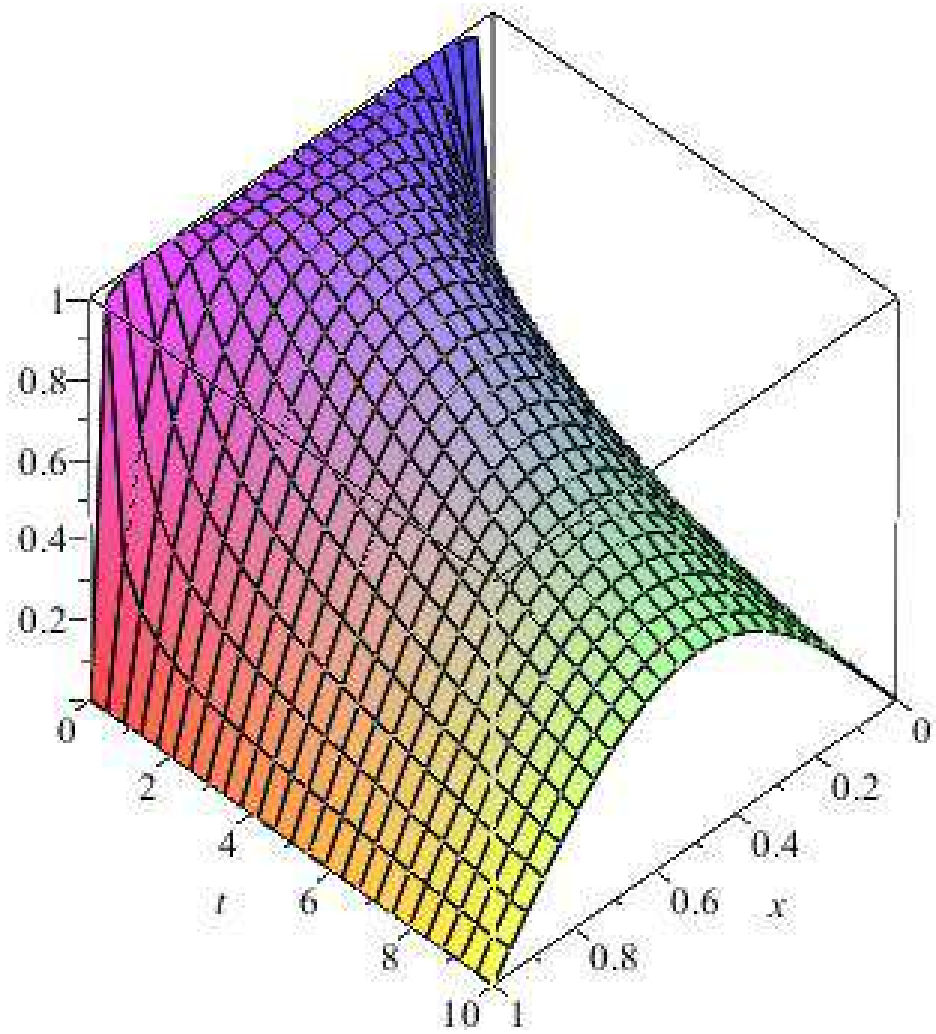
Distribution 1



Distribution 2



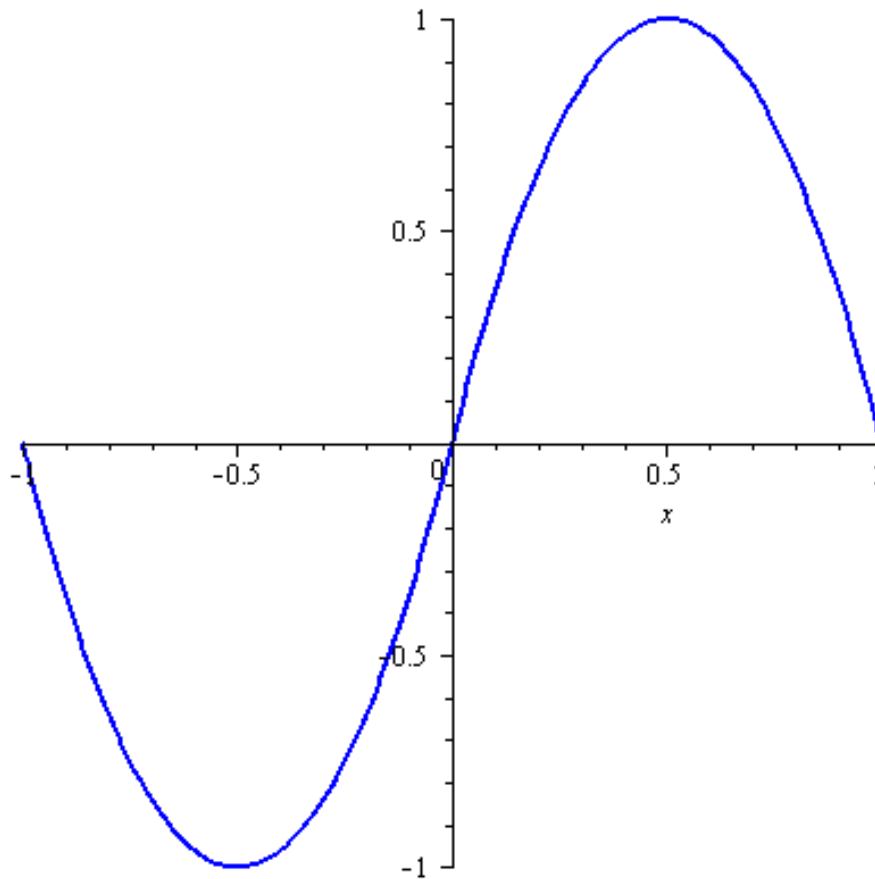
Distribution 3



Appendices 4: Maple worksheet 1

```
> alias(u=Heaviside) :
> f:=x->(-4*x*(-1-x))*(u(x+1)-u(x))+(4*x*(1-x))*(u(x)-u(x-1)) ;
      f:=x→-4x(-1-x)(u(x+1)-u(x))+4x(1-x)(u(x)-u(x
      -1))

> plot(f(x),x=-1..1,colour=blue,thickness=2) ;
```



```
> T:=2:
> bn:=2/T*Int(f(x)*sin(2*Pi*n*x/T),x=-
T/2..T/2)=2/T*int(f(x)*sin(2*Pi*n*x/T),x=-T/2..T/2) ;
```

$$b_n := \int_{-1}^1 (-4x(-1-x)(u(x+1)-u(x)) + 4x(1-x)(u(x)-u(x-1))) \sin(\pi n x) \, dx = \frac{-8 \sin(\pi n) \pi n - 16 \cos(\pi n) + 16}{\pi^3 n^3}$$

```

> N:=10:
> b:=array([],0..N):
> for n from 1 to N do
> b[n]:=2/T*integrate(f(x)*sin(2*n*Pi*x/T),x=-T/2..T/2);
> end do;

```

$$b_1 := \frac{32}{\pi^3}$$

$$b_2 := 0$$

$$b_3 := \frac{32}{27\pi^3}$$

$$b_4 := 0$$

$$b_5 := \frac{32}{125\pi^3}$$

$$b_6 := 0$$

$$b_7 := \frac{32}{343\pi^3}$$

$$b_8 := 0$$

$$b_9 := \frac{32}{729\pi^3}$$

$$b_{10} := 0$$

```

> n:='n':
> F:=x->sum(b[n]*sin(2*n*Pi*x/T),n=1..N);

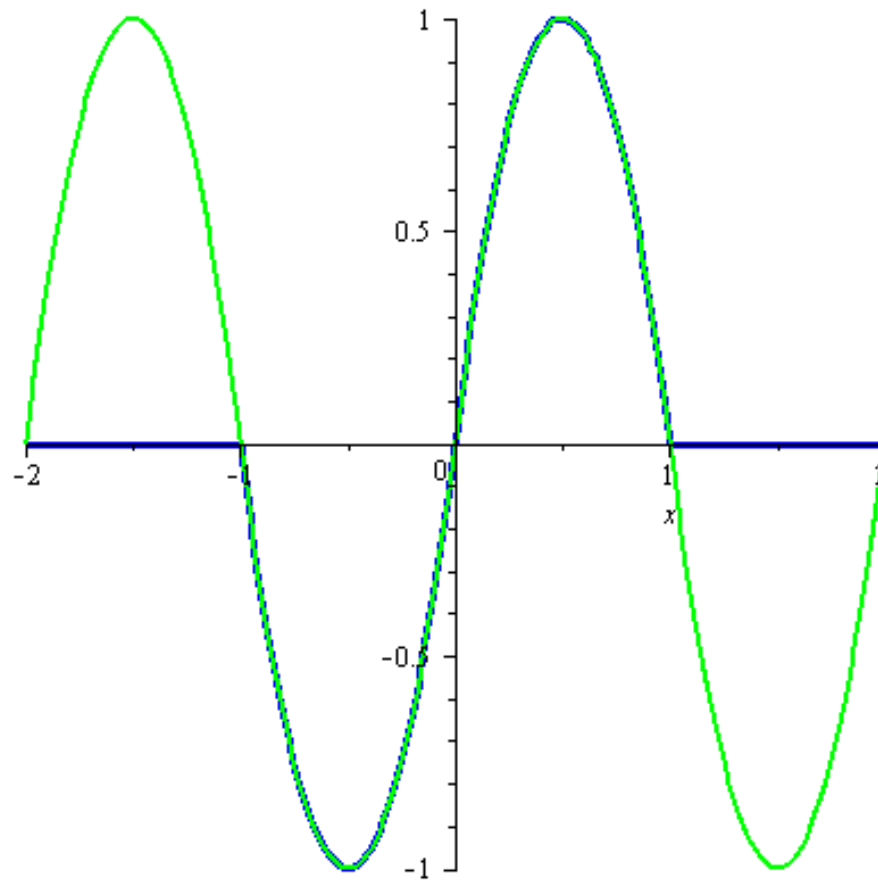
```

$$F := x \rightarrow \sum_{n=1}^N b_n \sin\left(\frac{2n\pi x}{T}\right)$$

```

> plot([f(x),F(x)],x=-
2..2,colour=[blue,green],thickness=[3,2]);

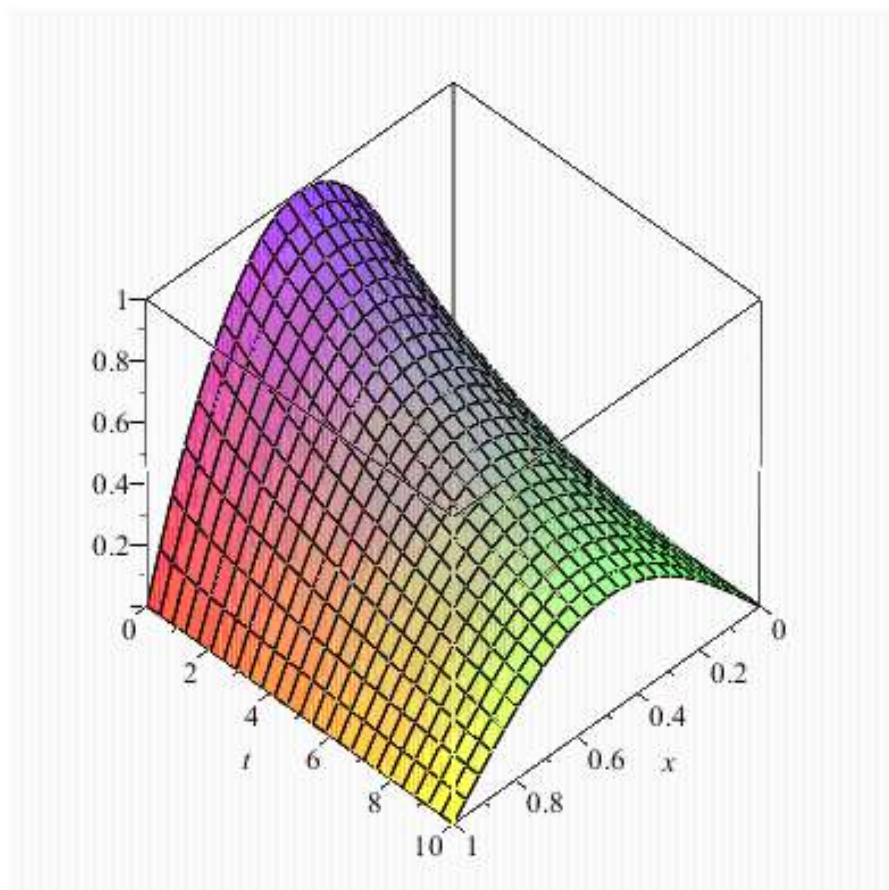
```



```
> k:=100:
> t:='t':
> phi:=(x,t)->sum(b[n]*sin(n*Pi*x)*exp(-(n*Pi)^2*t/k),n=1..N);
```

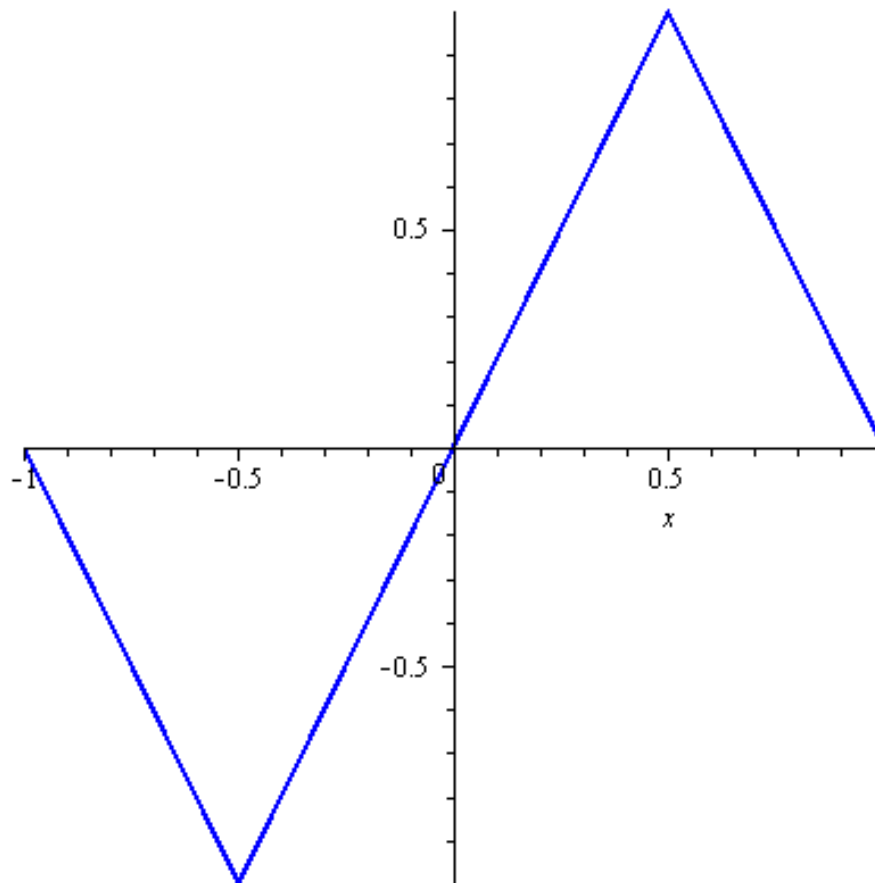
$$\phi := (x, t) \rightarrow \sum_{n=1}^N b_n \sin(n\pi x) e^{-\frac{\pi^2 n^2 t}{k}}$$

```
> plot3d(phi(x,t),x=0..1,t=0..10,axes=boxed);
```



Appendices 5: Maple worksheet 2

```
> alias(u=Heaviside):
> f:=x->(-2-2*x)*(u(x+1)-u(x+0.5))+(2*x)*(u(x+0.5)-u(x-0.5))+(2-2*x)*(u(x-0.5)-u(x-1));
f:=x->(-2-2x)(u(x+1)-u(x+0.5))+2x(u(x+0.5)-u(x-0.5))+(2-2x)(u(x-0.5)-u(x-1))
> plot(f(x),x=-1..1,colour=blue,thickness=2);
```



```
> T:=2:
> bn:=2/T*Int(f(x)*sin(2*Pi*n*x/T),x=-T/2..T/2)=2/T*int(f(x)*sin(2*Pi*n*x/T),x=-T/2..T/2);
```

$$\begin{aligned}
 b_n &:= \int_{-1}^1 ((-2-2x)(u(x+1)-u(x+0.5))+2x(u(x+0.5)-u(x-0.5))+(2-2x)(u(x-0.5)-u(x-1))) \sin(\pi n x) \\
 &\quad dx \\
 &= \frac{1}{n^2} (0.1013211836 (-4. \sin(3.141592654 n) \\
 &\quad + 8. \sin(1.570796327 n)))
 \end{aligned}$$

```

> N:=100:
> b:=array([],0..N):
> for n from 1 to N do
> b[n]:=2/T*integrate(f(x)*sin(2*n*Pi*x/T),x=-T/2..T/2);
> end do:
> n:='n':
> F:=x->sum(b[n]*sin(2*n*Pi*x/T),n=1..N);

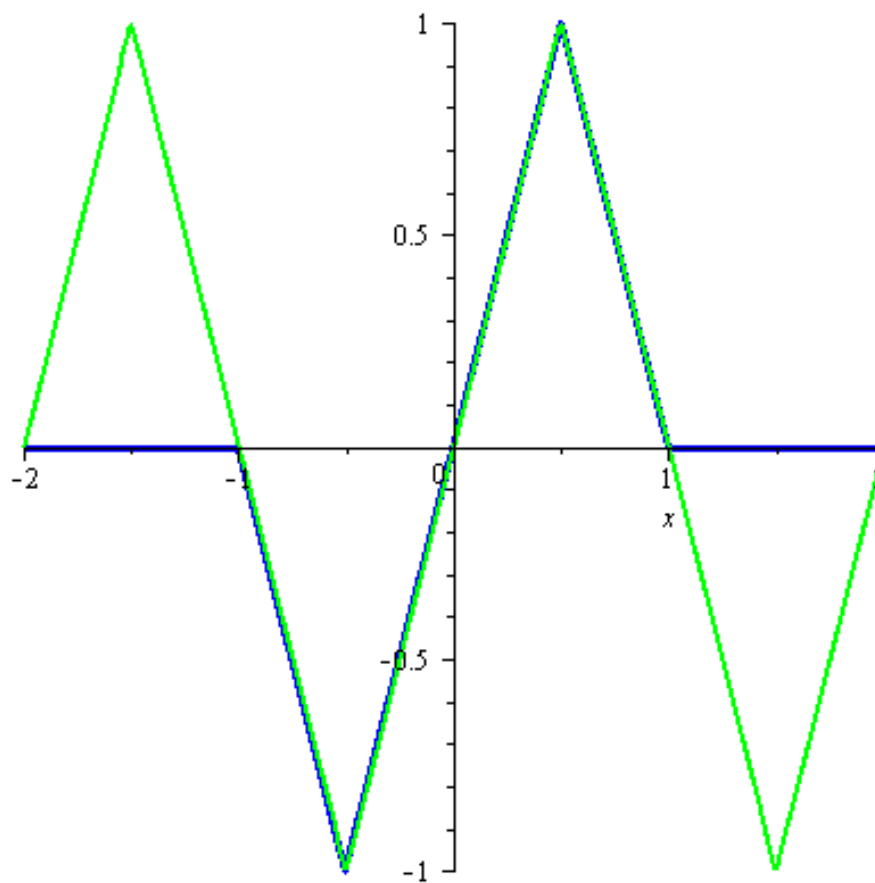
```

$$F := x \rightarrow \sum_{n=1}^N b_n \sin\left(\frac{2n\pi x}{T}\right)$$

```

> plot([f(x),F(x)],x=-
2..2,colour=[blue,green],thickness=[3,2]);

```



```

> k:=100:
> t:='t':
> phi:=(x,t)->sum(b[n]*sin(n*Pi*x)*exp(-(n*Pi)^2*t/k),n=1..N);

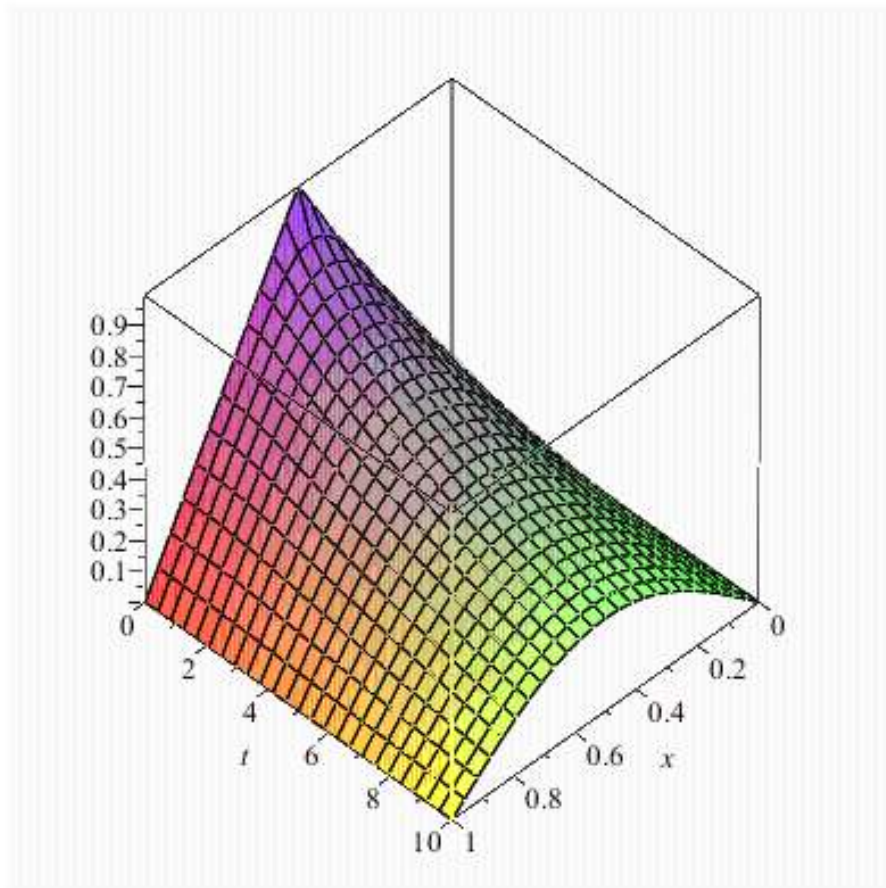
```

$$\phi := (x, t) \rightarrow \sum_{n=1}^N b_n \sin(n\pi x) e^{-\frac{\pi^2 n^2 t}{k}}$$

```

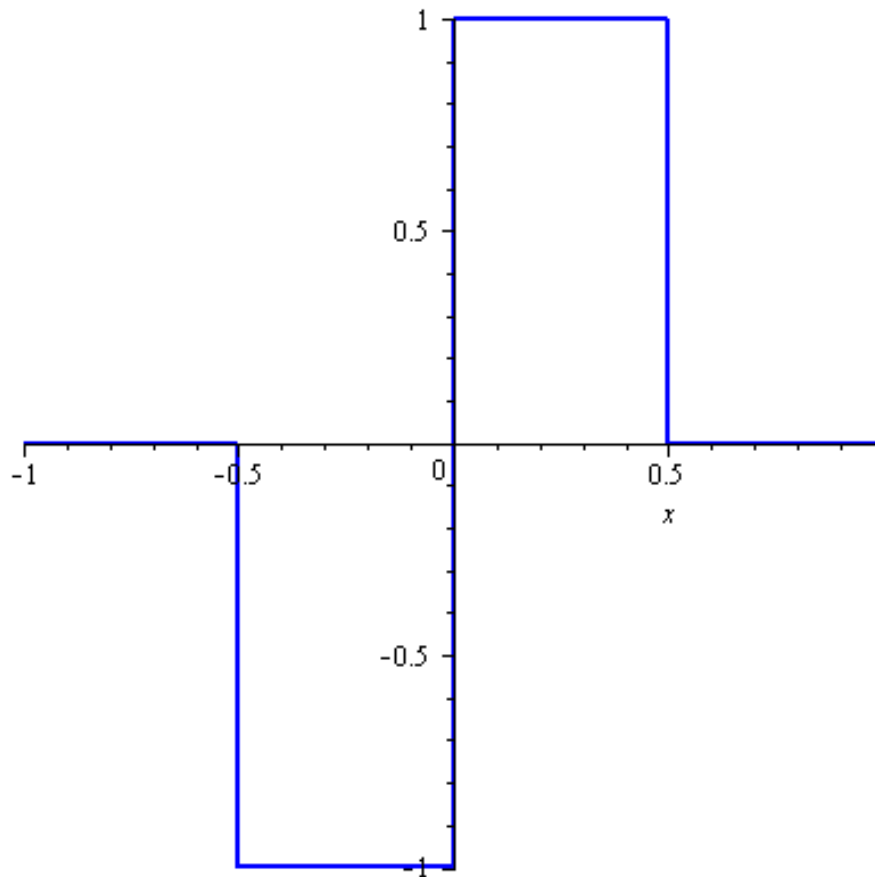
> plot3d(phi(x,t),x=0..1,t=0..10,axes=boxed);

```



Appendices 6: Maple worksheet 3

```
> alias(u=Heaviside) :
> f:=x->u(x)-u(x-0.5)+u(x)-u(x+0.5) ;
      f:=x→2u(x)-u(x-0.5)-u(x+0.5)
> plot(f(x),x=-1..1,colour=blue,thickness=2) ;
```



```
> T:=1:
> bn:=2/T*Int(f(x)*sin(2*Pi*n*x/T),x=-
T/2..T/2)=2/T*int(f(x)*sin(2*Pi*n*x/T),x=-T/2..T/2) ;
```

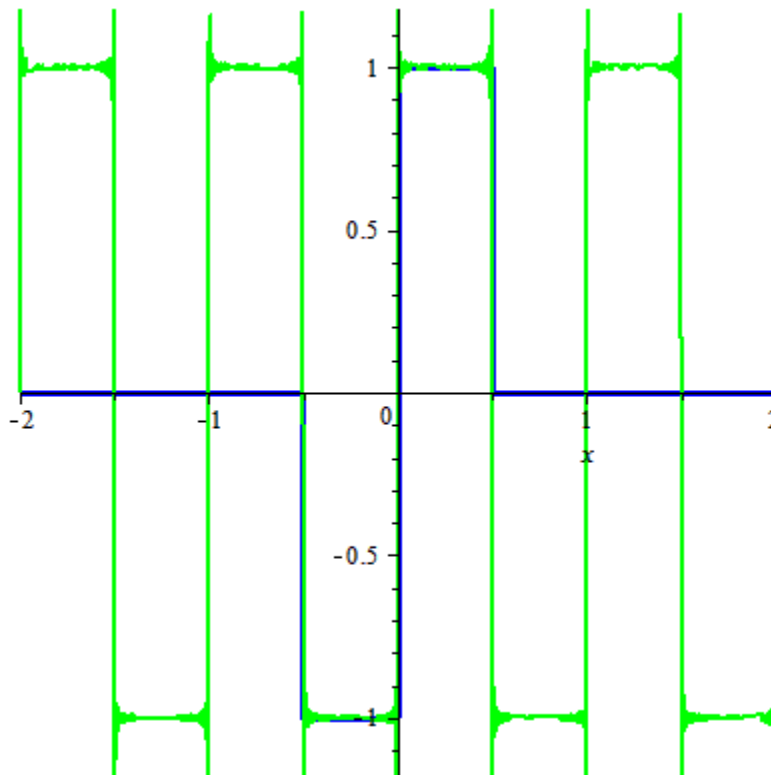
$$bn := 2 \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} (2u(x) - u(x-0.5) - u(x+0.5)) \sin(2\pi nx) dx \right)$$

$$= \frac{0.6366197724 (-1.\cos(3.141592654n) + 1.)}{n}$$

```
> N:=150:
> b:=array([],0..N):
> for n from 1 to N do
> b[n]:=2/T*integrate(f(x)*sin(2*n*Pi*x/T),x=-T/2..T/2) ;
> end do:
> n:='n':
> F:=x->sum(b[n]*sin(2*n*Pi*x/T),n=1..N) ;
>
```

$$F := x \rightarrow \sum_{n=1}^N b_n \sin\left(\frac{2n\pi x}{T}\right)$$

```
> plot([f(x), F(x)], x=-
2..2, colour=[blue, green], thickness=[3, 2]);
```



```
> k:=100:
```

```
> t:='t':
```

```
> phi:=(x,t)->sum(b[n]*sin(n*Pi*x)*exp(-(n*Pi)^2*t/k), n=1..N);
```

$$\phi := (x, t) \rightarrow \sum_{n=1}^N b_n \sin(n\pi x) e^{-\frac{\pi^2 n^2 t}{k}}$$

```
> plot3d(phi(x,t), x=0..1, t=0..10, axes=boxed);
```

