

Complex Form of the Fourier Series

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Write $\frac{1}{2}a_0 = c_0$

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Write $\frac{1}{2}a_0 = c_0$ $\frac{1}{2}(a_n - ib_n) = c_n$

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$$f(x) = c_0 + \sum_{n=1}^{\infty} [c_n e^{inx} + k_n e^{-inx}]$$

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The coefficients are given by

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$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)(\cos nx - i \sin nx) dx$$

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For a function of general period T

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2n\pi x/T}$$

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For a function of general period T

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2n\pi x/T}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-i2n\pi x/T} dx, \quad n = 0, \pm 1, \pm 2, \dots$$

Odd and Even Functions

$$\begin{aligned}c_n &= \frac{1}{T} \int_{-T/2}^{T/2} f(x) \left(\cos\left(\frac{2n\pi x}{T}\right) - i \sin\left(\frac{2n\pi x}{T}\right) \right) dx \\&= \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2n\pi x}{T}\right) dx - \frac{i}{T} \int_{-T/2}^{T/2} f(x) \sin\left(\frac{2n\pi x}{T}\right) dx\end{aligned}$$

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If $f(x)$ is **even**, then $f(x) \sin\left(\frac{2n\pi x}{T}\right)$ is **odd**, and so

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In this case, $c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2n\pi x}{T}\right) dx$

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If $f(x)$ is **even**, then $f(x) \sin\left(\frac{2n\pi x}{T}\right)$ is **odd**, and so

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In this case, $c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2n\pi x}{T}\right) dx$ So c_n is **real**

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2n\pi x}{T}\right) dx$$

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and so

$$c_{-n} = \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2(-n)\pi x}{T}\right) dx$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2n\pi x}{T}\right) dx$$

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$$c_{-n} = \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2(-n)\pi x}{T}\right) dx$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2n\pi x}{T}\right) dx \quad (\text{cosine is even})$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2n\pi x}{T}\right) dx$$

and so

$$\begin{aligned}
 c_{-n} &= \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2(-n)\pi x}{T}\right) dx \\
 &= \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2n\pi x}{T}\right) dx && \text{(cosine is even)} \\
 &= c_n
 \end{aligned}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2n\pi x/T}$$

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$$= c_0 + \sum_{n=1}^{\infty} c_n e^{i2n\pi x/T} + \sum_{n=-\infty}^{-1} c_n e^{i2n\pi x/T}$$

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 f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{i2n\pi x/T} \\
 &= c_0 + \sum_{n=1}^{\infty} c_n e^{i2n\pi x/T} + \sum_{n=-\infty}^{-1} c_n e^{i2n\pi x/T} \\
 &= c_0 + \sum_{n=1}^{\infty} c_n e^{i2n\pi x/T} + \sum_{n=\infty}^1 c_{-n} e^{-i2n\pi x/T} \\
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&= c_0 + \sum_{n=1}^{\infty} c_n e^{i2n\pi x/T} + \sum_{n=\infty}^1 c_n e^{-i2n\pi x/T} \\
&= c_0 + \sum_{n=1}^{\infty} c_n \left(e^{i2n\pi x/T} + e^{-i2n\pi x/T} \right)
\end{aligned}$$

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&= c_0 + \sum_{n=1}^{\infty} c_n \left(e^{i2n\pi x/T} + e^{-i2n\pi x/T} \right) \\
&= c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{2n\pi x}{T}\right)
\end{aligned}$$

Odd and Even Functions

$$\begin{aligned}c_n &= \frac{1}{T} \int_{-T/2}^{T/2} f(x) \left(\cos\left(\frac{2n\pi x}{T}\right) - i \sin\left(\frac{2n\pi x}{T}\right) \right) dx \\&= \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2n\pi x}{T}\right) dx - \frac{i}{T} \int_{-T/2}^{T/2} f(x) \sin\left(\frac{2n\pi x}{T}\right) dx\end{aligned}$$

If $f(x)$ is **odd**, then $f(x) \cos\left(\frac{2n\pi x}{T}\right)$ is **odd**, and so

$$\int_{-T/2}^{T/2} f(x) \cos\left(\frac{2n\pi x}{T}\right) dx = 0$$

In this case, $c_n = \frac{-i}{T} \int_{-T/2}^{T/2} f(x) \sin\left(\frac{2n\pi x}{T}\right) dx$ So c_n is **imaginary**

$$c_n = \frac{-i}{T} \int_{-T/2}^{T/2} f(x) \sin\left(\frac{2n\pi x}{T}\right) dx$$

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$$= \frac{-i}{T} \int_{-T/2}^{T/2} f(x) \cdot -\sin\left(\frac{2n\pi x}{T}\right) dx \quad (\text{sine is odd})$$

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and so

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&= \sum_{n=1}^{\infty} c_n i \sin\left(\frac{2n\pi x}{T}\right)
\end{aligned}$$

In summary, if

$f(x)$ is $\begin{Bmatrix} \text{odd} \\ \text{even} \end{Bmatrix}$, then

the coefficients are $\begin{Bmatrix} \text{real} \\ \text{imaginary} \end{Bmatrix}$

Also $c_{-n} = \begin{Bmatrix} c_n \\ -c_n \end{Bmatrix}$

and only $\begin{Bmatrix} \text{cosine} \\ \text{sine} \end{Bmatrix}$ terms remain .

Example 1

Show that the complex Fourier series of the square wave of period 4 given by

$$f(x) = \begin{cases} 0 & \text{when } -2 < x < 0 \\ 1 & \text{when } 0 < x < 2. \end{cases}$$

is

$$f(x) = \frac{1}{2} - \frac{i}{2\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1 - (-1)^n}{n} e^{in\pi x/2}$$

Convert the series to real form.

The complex series is $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2n\pi x/T}$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-i2n\pi x/T} dx, \quad n = 0, \pm 1, \pm 2, \dots$$

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$$\begin{aligned}
c_n &= \frac{1}{4} \int_{-2}^2 f(x) dx \\
&= \frac{1}{4} \left(\int_{-2}^0 0 \cdot e^{-i2n\pi x/T} dx + \int_0^2 1 \cdot e^{-i2n\pi x/T} dx \right) \\
&= \frac{1}{4} \left(0 + \int_0^2 1 \cdot e^{-in\pi x/2} dx \right) \\
&= \frac{1}{4} \cdot \frac{-2}{in\pi} \left[e^{-in\pi x/2} \right]_0^2 \\
&= \frac{-1}{2in\pi} (e^{-in\pi} - 1) \\
&= \frac{i}{2n\pi} (e^{-in\pi} - 1)
\end{aligned}$$

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$$\begin{aligned} c_n e^{in\pi x/2} + c_{-n} e^{-in\pi x/2} = & \frac{-i}{2n\pi} \left(1 - (-1)^n\right) \left(\cos \frac{n\pi x}{2} + i \sin \frac{n\pi x}{2}\right) \\ & + \frac{i}{2n\pi} \left(1 - (-1)^{-n}\right) \left(\cos \frac{n\pi x}{2} - i \sin \frac{n\pi x}{2}\right) \end{aligned}$$

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