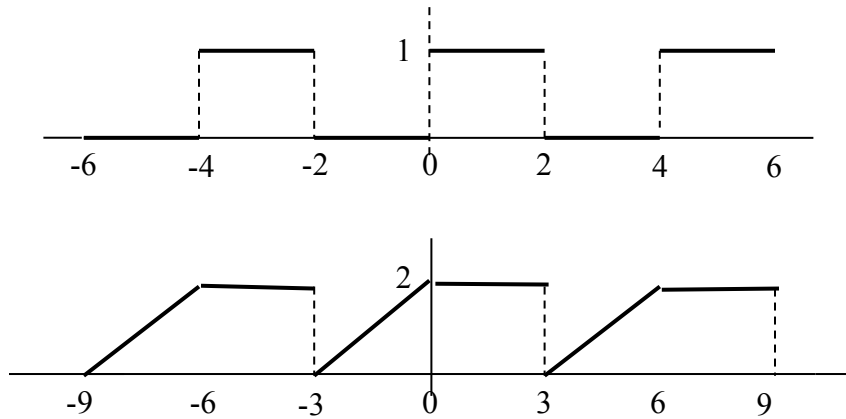


1. Fourier Series

1. Introduction

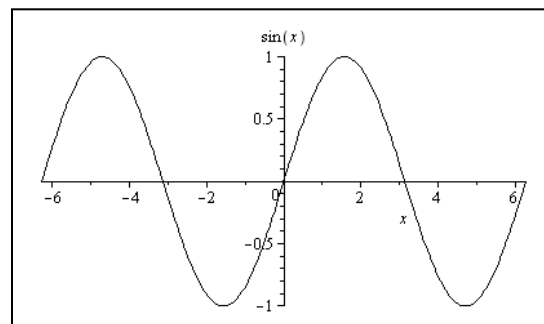
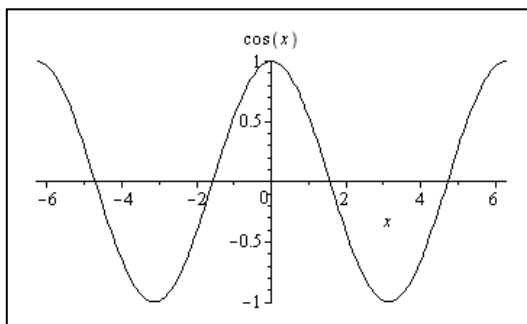
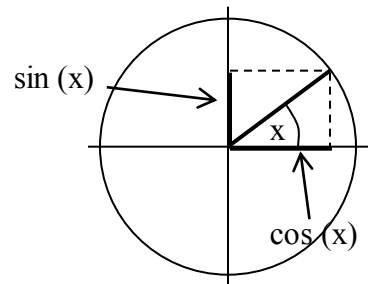
Many signals consist of periodic waveforms such as those shown below.



In the 1820's Joseph Fourier showed that any periodic function can be represented as a set of harmonics of the basic periodic functions given by sine and cosine. This discovery provides the basis of Fourier techniques underlying modern digital signal processing, coding and information theory.

2. Sine and Cosine Functions

Recall the functions $\sin(x)$ and $\cos(x)$. They can be thought of as the projections onto the vertical and horizontal axes respectively of the unit radius of a circle when the radius makes an angle x with the horizontal.



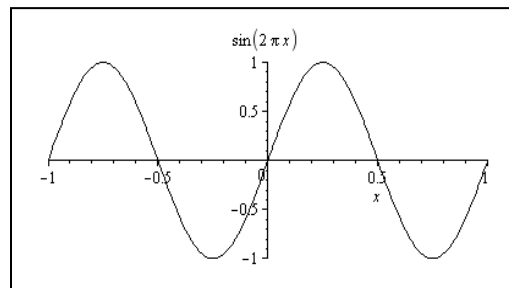
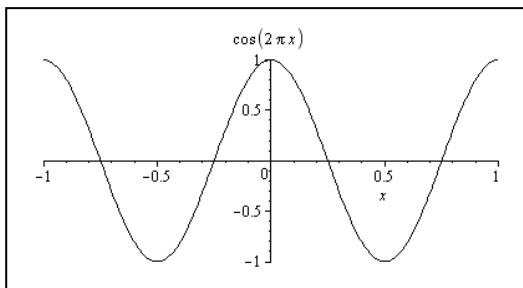
Outside of the range $[0, 2\pi]$ these functions repeat themselves, so that $\sin(x+2\pi)=\sin(x)$, $\sin(x+4\pi)=\sin(x)$ etc. These functions are periodic with period 2π .

The following values of the sine and cosine functions are important and should be remembered:

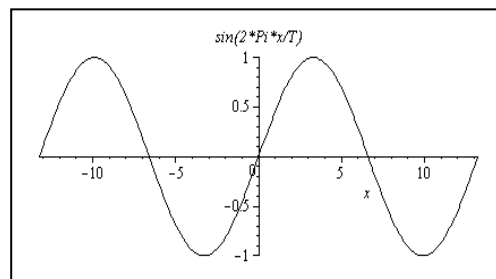
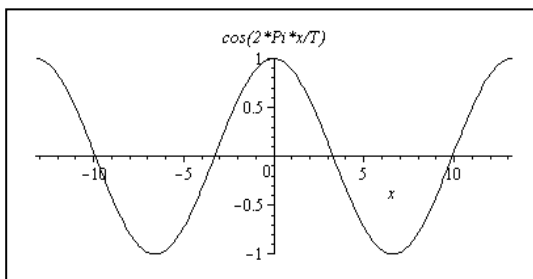
x	0	$\pi/2$	$n\pi$	$2n\pi$
$\cos(x)$	1	0	$(-1)^n$	1
$\sin(x)$	0	1	0	0

Sine and cosine can be defined on other intervals with different periods.

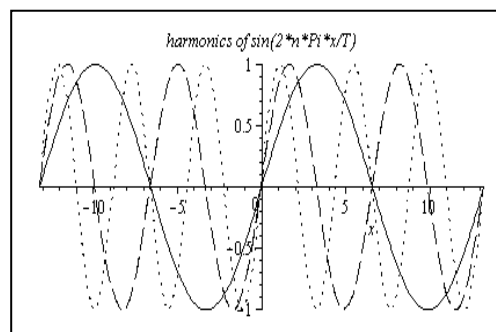
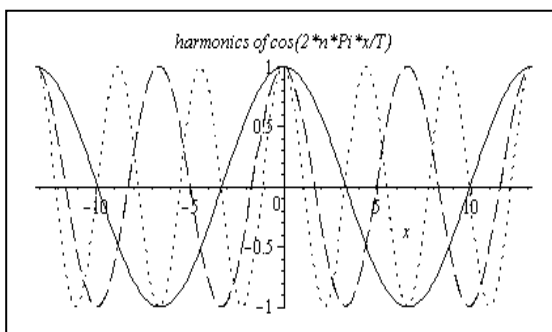
For example, the functions $\cos(2\pi x)$ and $\sin(2\pi x)$ are both periodic with period 1.



The functions $\cos\left(\frac{2\pi x}{T}\right)$ and $\sin\left(\frac{2\pi x}{T}\right)$ are periodic with period T . Eg for $T=13.2$



The harmonics $\cos\left(\frac{2n\pi x}{T}\right)$ and $\sin\left(\frac{2n\pi x}{T}\right)$ are periodic with period T/n . Setting $n=1$ gives the first harmonic (or fundamental), $n=2$ the second harmonic etc.



3. Full-Range Fourier Series

Any periodic function $f(x)$, of period T , can be represented as a sum of harmonics of sine and/or cosine. Mathematically this is written as follows:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi x}{T}\right) + b_n \sin\left(\frac{2n\pi x}{T}\right) \right]$$

where a_n is the coefficient of each cosine harmonic and b_n is the coefficient of each sine harmonic. The coefficients a_n and b_n are found using the following formulae:

$$a_0 = \frac{2}{T} \int_0^T f(x) dx, \quad a_n = \frac{2}{T} \int_0^T f(x) \cos \frac{2n\pi x}{T} dx \quad n = 1, 2, 3, \dots$$

and

$$b_n = \frac{2}{T} \int_0^T f(x) \sin \frac{2n\pi x}{T} dx \quad n = 1, 2, 3, \dots$$

Notes

- The series is periodic with period T
- The interval 0 to T can be replaced by $-\frac{T}{2}$ to $\frac{T}{2}$ if this is more convenient.
- If $f(x)$ is discontinuous so that it has two possible values (that is it has a jump) then the series converges (as the number of terms increases) to the average of the two values.

Proof of the Fourier coefficients

The derivation of the Fourier coefficients above relies on the following orthogonality relations for the sine and cosine functions. With $\omega = 2\pi/T$ these are:

$$\int_0^T \cos(n\omega x) dx = \begin{cases} 0 & n \neq 0 \\ T & n = 0 \end{cases}$$

$$\int_0^T \sin(n\omega x) dx = 0 \quad (\text{all } n)$$

$$\int_0^T \sin(m\omega x) \sin(n\omega x) dx = \begin{cases} 0 & m \neq n \\ T/2 & m = n \neq 0 \end{cases}$$

$$\int_0^T \cos(m\omega x) \cos(n\omega x) dx = \begin{cases} 0 & m \neq n \\ T/2 & m = n \neq 0 \end{cases}$$

$$\int_0^T \cos(m\omega x) \sin(n\omega x) dx = 0 \quad (\text{all } n).$$

To find a_0

Integrating the series term by term over the interval $[0, T]$, it is found that each term on the right hand side is zero apart from the term involving a_0 . That is,

$$\int_0^T f(x) dx = \frac{1}{2} a_0 \int_0^T dx + \sum_{n=1}^{\infty} \left[a_n \int_0^T \cos(n\omega x) dx + b_n \int_0^T \sin(n\omega x) dx \right]$$

$$\int_0^T f(x) dx = \frac{1}{2} a_0 \int_0^T dx + \sum_{n=1}^{\infty} [a_n 0 + b_n 0].$$

Hence
$$a_0 = \frac{2}{T} \int_0^T f(x) dx.$$

(This implies that $\frac{1}{2} a_0$ is the mean value of the function – a useful check).

To find a_n

Integrate the series term by term again, but multiply each term by $\cos(m\omega x)$ first.

$$\int_0^T f(x) \cos(m\omega x) dx = \frac{1}{2} a_0 \int_0^T \cos(m\omega x) dx + \sum_{n=1}^{\infty} a_n \int_0^T \cos(m\omega x) \cos(n\omega x) dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_0^T \cos(m\omega x) \sin(n\omega x) dx$$

Each term on the right is zero except the term in the first sum when $n=m$. Hence

$$\int_0^T f(x) \cos(m\omega x) dx = a_m \frac{T}{2}$$

or

$$a_m = \frac{2}{T} \int_0^T f(x) \cos(m\omega x) dx.$$

Replacing m by n gives the required result.

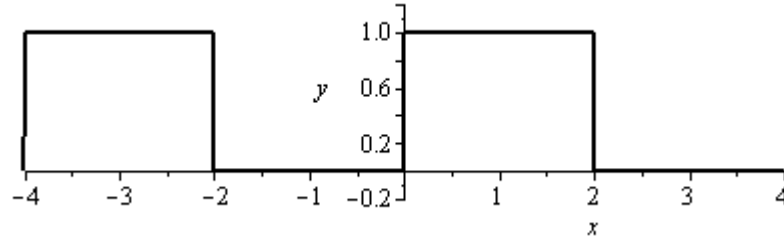
To find b_n

This is a similar but each term is multiplied by $\sin(m\omega x)$ first. It is left as an exercise for the student to complete.

Example – Square Wave

Find the Fourier series of the square wave of period 4 given by

$$f(x) = \begin{cases} 0 & \text{when } -2 < x < 0 \\ 1 & \text{when } 0 < x < 2. \end{cases}$$



It makes little difference in this case whether we use $[0, T]$ or $[-T/2, T/2]$. Choosing the latter and finding the a_n first, we have

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(x) dx = \frac{2}{4} \int_{-2}^2 f(x) dx = \frac{1}{2} \left(\int_{-2}^0 0 dx + \int_0^2 1 dx \right) = \frac{1}{2} [x]_0^2 = \frac{1}{2} \times 2 = 1,$$

and

$$\begin{aligned} a_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos \frac{2n\pi x}{T} dx = \frac{2}{4} \int_{-2}^2 f(x) \cos \frac{2n\pi x}{4} dx \\ &= \frac{1}{2} \left(\int_{-2}^0 0 \cdot \cos \frac{n\pi x}{2} dx + \int_0^2 1 \cdot \cos \frac{n\pi x}{2} dx \right) \\ &= \frac{1}{2} \frac{2}{n\pi} \left[\sin \frac{n\pi x}{2} \right]_0^2 = \frac{1}{n\pi} [\sin(n\pi) - \sin(0)] = 0. \end{aligned}$$

Thus all the cosine terms in this series are zero. The coefficients of the sine terms, on the other hand, are given by the b_n , which are found as follows:

$$\begin{aligned} b_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(x) \sin \frac{2n\pi x}{T} dx = \frac{2}{4} \int_{-2}^2 f(x) \sin \frac{2n\pi x}{4} dx \\ &= \frac{1}{2} \left(\int_{-2}^0 0 \cdot \sin \frac{n\pi x}{2} dx + \int_0^2 1 \cdot \sin \frac{n\pi x}{2} dx \right) \\ &= \frac{1}{2} \frac{2}{n\pi} \left[-\cos \frac{n\pi x}{2} \right]_0^2 = \frac{1}{n\pi} [-\cos(n\pi) - (-1)] = \frac{1 - (-1)^n}{n\pi}. \end{aligned}$$

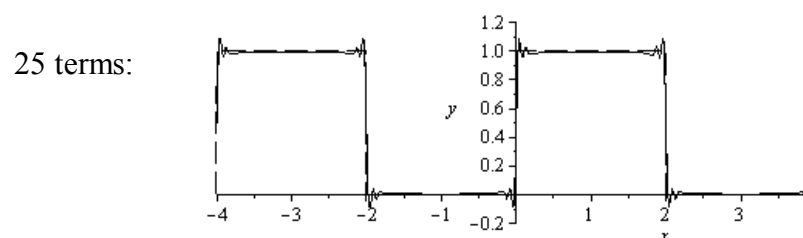
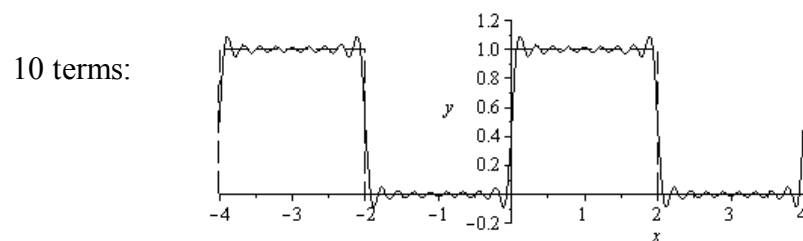
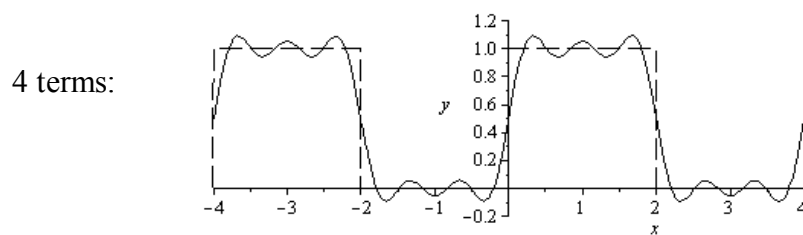
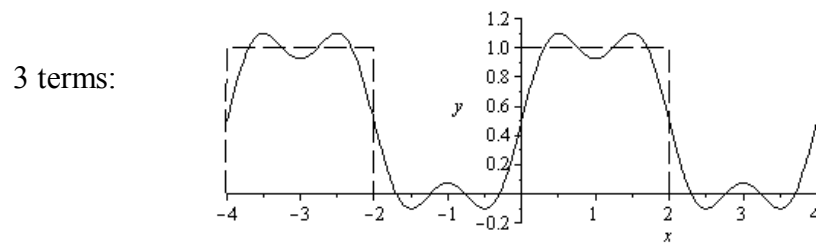
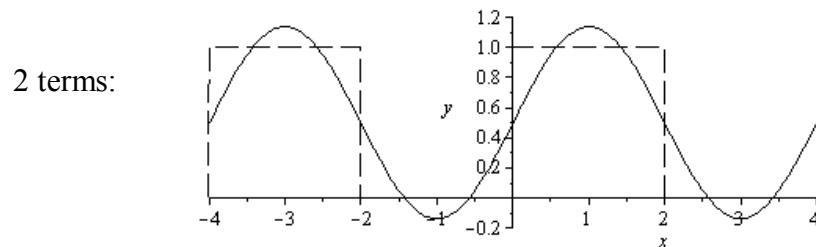
Thus the series is $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n\pi} \sin\left(\frac{2n\pi x}{4}\right)$

or $\frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin\left(\frac{n\pi x}{2}\right).$

Note that the terms in the sum are zero for even values of n . Hence the first few non-zero terms in the series are:

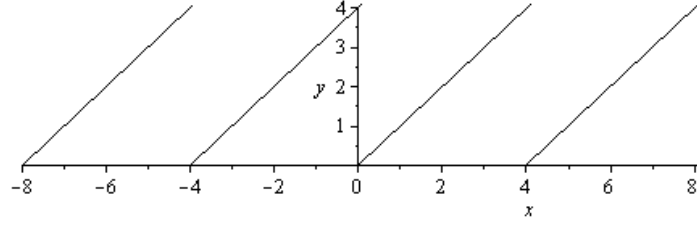
$$\frac{1}{2} + \frac{2}{\pi} \left[\sin\left(\frac{\pi x}{2}\right) + \frac{1}{3} \sin\left(\frac{3\pi x}{2}\right) + \frac{1}{5} \sin\left(\frac{5\pi x}{2}\right) + \dots \right].$$

The series converges to the original square wave as more harmonics are included, that is, as n increases. This is shown as follows:



Example - Saw Tooth

Find the Fourier series of the saw tooth wave of period 4 given by the function $f(x) = x$ for $0 \leq x < 4$.



In this example it is more convenient to integrate over $[0, T]$ rather than $[-T/2, T/2]$. Hence the coefficients are found as:

$$a_0 = \frac{2}{T} \int_0^T f(x) dx = \frac{2}{4} \int_0^4 x dx = \frac{1}{2} \left[x^2 / 2 \right]_0^4 = \frac{1}{2} \times (8 - 0) = 4.$$

$$\begin{aligned} a_n &= \frac{2}{T} \int_0^T f(x) \cos \frac{2n\pi x}{T} dx = \frac{2}{4} \int_0^4 x \cos \frac{2n\pi x}{4} dx = \frac{1}{2} \int_0^4 x \cos \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \left[\left[x \frac{2}{n\pi} \sin \frac{n\pi x}{2} \right]_0^4 - \int_0^4 1 \cdot \frac{2}{n\pi} \sin \frac{n\pi x}{2} dx \right] \\ &= \frac{1}{2} \left[\left(\frac{8}{n\pi} \sin 2n\pi - 0 \right) + \left[\left(\frac{2}{n\pi} \right)^2 \cos \frac{n\pi x}{2} \right]_0^4 \right] \\ &= \frac{1}{2} \left[(0 - 0) + \frac{4}{n^2 \pi^2} \cos(2n\pi) - \frac{4}{n^2 \pi^2} \cos(0) \right] = 0. \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{T} \int_0^T f(x) \sin \frac{2n\pi x}{T} dx = \frac{2}{4} \int_0^4 x \sin \frac{2n\pi x}{4} dx = \frac{1}{2} \int_0^4 x \sin \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \left[\left[x \frac{-2}{n\pi} \cos \frac{n\pi x}{2} \right]_0^4 - \int_0^4 1 \cdot \frac{-2}{n\pi} \cos \frac{n\pi x}{2} dx \right] \\ &= \frac{1}{2} \left[\left(\frac{-8}{n\pi} \cos 2n\pi - 0 \right) + \left[\left(\frac{2}{n\pi} \right)^2 \sin \frac{n\pi x}{2} \right]_0^4 \right] \\ &= \frac{1}{2} \left[\frac{-8}{n\pi} + \left(\frac{2}{n\pi} \right)^2 \sin(2n\pi) - \left(\frac{2}{n\pi} \right)^2 \sin(0) \right] \\ &= \frac{-4}{n\pi} \end{aligned}$$

Thus the series is

$$2 + \sum_{n=1}^{\infty} \frac{-4}{n\pi} \sin\left(\frac{2n\pi x}{4}\right)$$

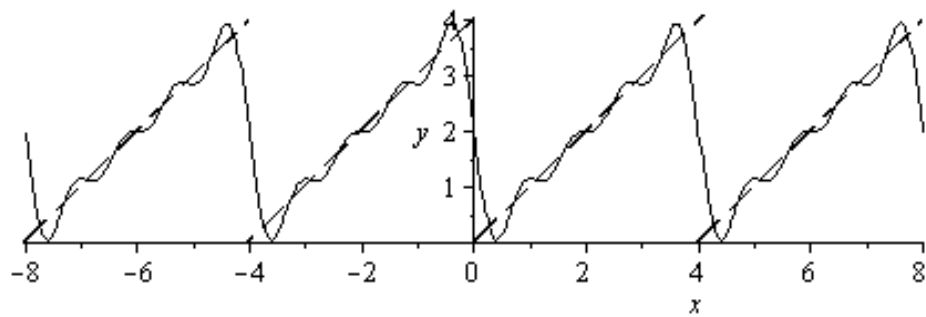
or

$$2 - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{2}\right)$$

The first five terms are:

$$2 - \frac{4}{\pi} \left(\frac{1}{1} \sin\left(\frac{\pi x}{2}\right) + \frac{1}{2} \sin\left(\frac{2\pi x}{2}\right) + \frac{1}{3} \sin\left(\frac{3\pi x}{2}\right) + \frac{1}{4} \sin\left(\frac{4\pi x}{2}\right) + \dots \right).$$

When plotted these give the following representation of the function:



Tutorial Exercises 1

1. A signal of period 0.5 is defined by $f(x) = \begin{cases} +2 & \text{for } -0.25 \leq x < 0 \\ -1 & \text{for } 0 \leq x < 0.25 \end{cases}$.

(a) Sketch $f(x)$ for $-1 \leq x \leq 1$.

(b) Show that the Fourier series expansion of $f(x)$ is $\frac{1}{2} - \frac{3}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin(4n\pi x)$.

(c) Write down the first four terms. Use them to estimate $f(0.1)$ to three decimal places and compare it with the exact value of $f(0.1)$.

2. An electro-mechanical device is controlled by a waveform defined over one period by

$$f(x) = \begin{cases} 0 & \text{when } -1 < x \leq 0 \\ -x & \text{when } 0 \leq x < 1 \end{cases}$$

(a) Sketch the graph of the function for the range $-3 < x < 3$.

(b) Show that the Fourier series is $-\frac{1}{4} + \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n^2 \pi^2} \cos(n\pi x) + \frac{(-1)^n}{n\pi} \sin(n\pi x) \right]$.

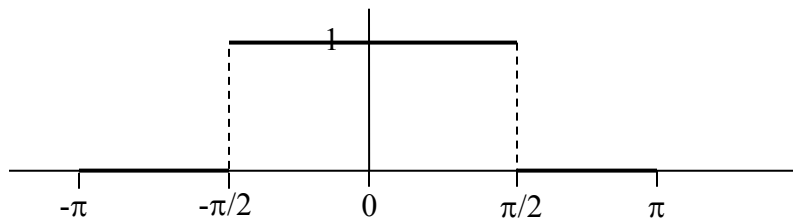
(c) Write down the first three terms (ie up to and including terms when $n=1$).

(d) State the value the series converges to when $x=1$.

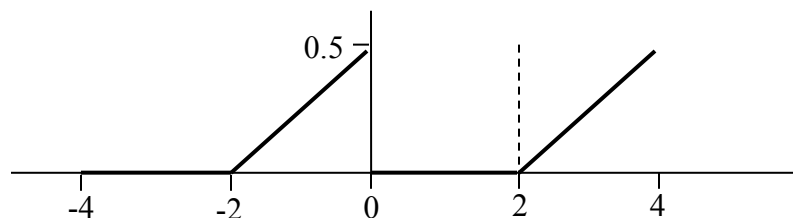
Homework Exercises 1

1. Find the Fourier series of the functions plotted below.

(i)



(ii)

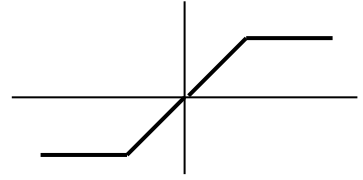


4. Odd and Even Functions

A function is said to be **odd** if $f(-x) = -f(x)$.

It then follows that $\int_{-a}^a f(x) dx = 0$.

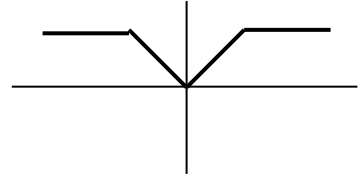
Note that $\sin(x)$ is an odd function.



A function is said to be **even** if $f(-x) = f(x)$.

It then follows that $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.

Note that $\cos(x)$ is an even function.



The following rules apply:

- (**even** function) $\times$ (**even** function) $\Rightarrow$ **even** function.
- (**odd** function) $\times$ (**odd** function) $\Rightarrow$ **even** function
- (**odd** function) $\times$ (**even** function) $\Rightarrow$ **odd** function

Fourier series for an odd function $f(x)$

Since the integrand for the cosine coefficients is an odd function then

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos \frac{2n\pi x}{T} dx = 0.$$

Hence the Fourier expansion consists of a series of sine terms only, and the sine integral can be 'doubled up' so that

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \sin \frac{2n\pi x}{T} dx = \frac{4}{T} \int_0^{T/2} f(x) \sin \frac{2n\pi x}{T} dx .$$

Fourier series for an even function

Since the integrand for the sine coefficients is an odd function then

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \sin \frac{2n\pi x}{T} dx = 0.$$

Hence the Fourier expansion consists of a series of cosine terms only, and the cosine integrals can be 'doubled up' so that

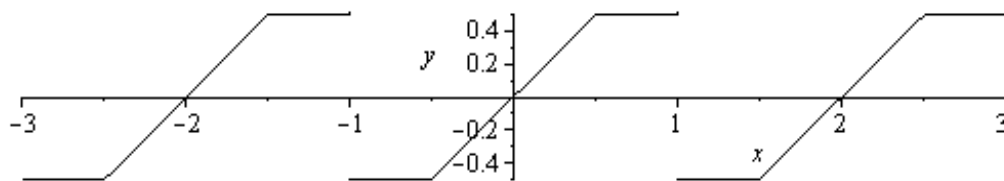
$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(x) dx = \frac{4}{T} \int_0^{T/2} f(x) dx.$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos \frac{2n\pi x}{T} dx = \frac{4}{T} \int_0^{T/2} f(x) \cos \frac{2n\pi x}{T} dx.$$

Odd Function Example

Consider the function defined over one period by

$$f(x) = \begin{cases} -1/2 & \text{when } -1 \leq x < -1/2 \\ x & \text{when } -1/2 \leq x < 1/2 \\ 1/2 & \text{when } 1/2 \leq x < 1 \end{cases}$$



This is an odd function, so $a_n=0$ ($n=0,1,2,3,\dots$) and the series contains sine terms only. The coefficients of the sine terms are given by:

$$\begin{aligned} b_n &= \frac{4}{T} \int_0^{T/2} f(x) \sin \frac{2n\pi x}{T} dx = \frac{4}{2} \int_0^1 f(x) \sin \frac{2n\pi x}{2} dx = 2 \int_0^1 f(x) \sin(n\pi x) dx \\ &= 2 \left(\int_0^{1/2} x \sin(n\pi x) dx + \int_{1/2}^1 \frac{1}{2} \sin(n\pi x) dx \right) \end{aligned}$$

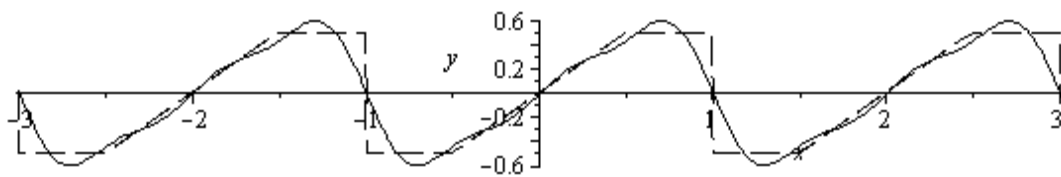
$$\begin{aligned}
&= 2 \left(\left[x \frac{-1}{n\pi} \cos(n\pi x) \right]_0^{1/2} - \int_0^{1/2} 1 \frac{-1}{n\pi} \cos(n\pi x) dx + \frac{1}{2} \frac{-1}{n\pi} [\cos(n\pi x)]_{1/2}^1 \right) \\
&= 2 \left(\frac{-1}{n\pi} \left(\frac{1}{2} \cos \frac{n\pi}{2} - 0 \right) + \left(\frac{1}{n\pi} \right)^2 [\sin(n\pi x)]_0^{1/2} + \frac{-1}{2n\pi} \left(\cos(n\pi) - \cos \frac{n\pi}{2} \right) \right) \\
&= 2 \left(\left(\frac{1}{n\pi} \right)^2 \left(\sin \frac{n\pi}{2} - \sin(0) \right) - \frac{(-1)^n}{2n\pi} \right) \\
&= \frac{2 \sin(n\pi/2)}{n^2 \pi^2} - \frac{(-1)^n}{n\pi}.
\end{aligned}$$

Thus the series is $\sum_{n=1}^{\infty} \left(\frac{2 \sin(n\pi/2)}{n^2 \pi^2} - \frac{(-1)^n}{n\pi} \right) \sin(n\pi x)$.

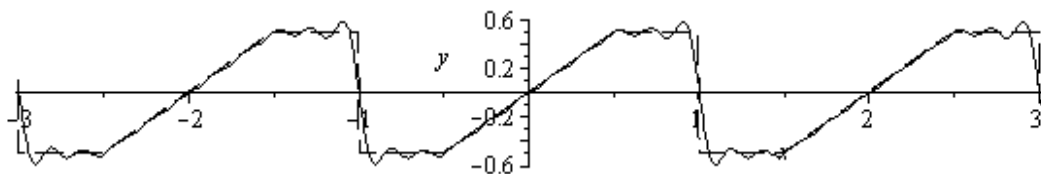
The first three terms in the series are:

$$\left(\frac{2}{\pi^2} + \frac{1}{\pi} \right) \sin(n\pi x) + \left(0 - \frac{1}{2\pi} \right) \sin(2\pi x) + \left(\frac{-2}{9\pi^2} + \frac{1}{3\pi} \right) \sin(3\pi x) + \dots$$

When plotted these three terms give the following representation of the function:



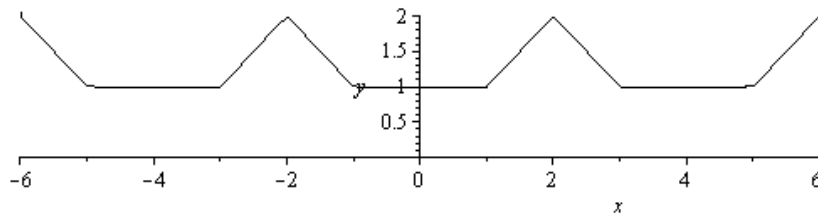
This is not particularly accurate, but increasing the number of terms to ten gives a better representation:



Even Function Example

Find the Fourier series of the wave of period 4 given by the function

$$f(x) = \begin{cases} -x & \text{when } -2 \leq x < -1 \\ 1 & \text{when } -1 \leq x < 1 \\ x & \text{when } 1 \leq x < 2 \end{cases}$$



This is an even function, so $b_n=0$ ($n=1,2,3,\dots$) and the series contains cosine terms only.

The coefficients of the cosine terms are given by:

$$a_0 = \frac{4}{T} \int_0^{T/2} f(x) dx = \frac{4}{4} \int_0^2 f(x) dx = \int_0^1 1 dx + \int_1^2 x dx = [x]_0^1 + \left[\frac{x^2}{2} \right]_1^2 = (1-0) + (2 - \frac{1}{2}) = \frac{5}{2}.$$

$$\begin{aligned} a_n &= \frac{4}{T} \int_0^{T/2} f(x) \cos \frac{2n\pi x}{T} dx = \frac{4}{4} \int_0^2 f(x) \cos \frac{2n\pi x}{4} dx \\ &= \int_0^1 1 \cos \frac{n\pi x}{2} dx + \int_1^2 x \cos \frac{n\pi x}{2} dx \\ &= \frac{2}{n\pi} \left[\sin \frac{n\pi x}{2} \right]_0^1 + \left[x \frac{2}{n\pi} \sin \frac{n\pi x}{2} \right]_1^2 - \int_1^2 1 \frac{2}{n\pi} \sin \frac{n\pi x}{2} dx \\ &= \frac{2}{n\pi} \left(\sin \frac{n\pi}{2} - \sin(0) \right) + \left(\frac{4}{n\pi} \sin(n\pi) - \frac{2}{n\pi} \sin \frac{n\pi}{2} \right) - \left(\frac{2}{n\pi} \right)^2 \left[-\cos \frac{n\pi x}{2} \right]_1^2 \\ &= \left(\frac{2}{n\pi} \right)^2 \left(\cos(n\pi) - \cos \frac{n\pi}{2} \right) = \left(\frac{2}{n\pi} \right)^2 \left((-1)^n - \cos \frac{n\pi}{2} \right) \end{aligned}$$

Thus the series is

$$\frac{5}{4} + \sum_{n=1}^{\infty} \left(\frac{2}{n\pi} \right)^2 \left((-1)^n - \cos \frac{n\pi}{2} \right) \cos \frac{n\pi x}{2}$$

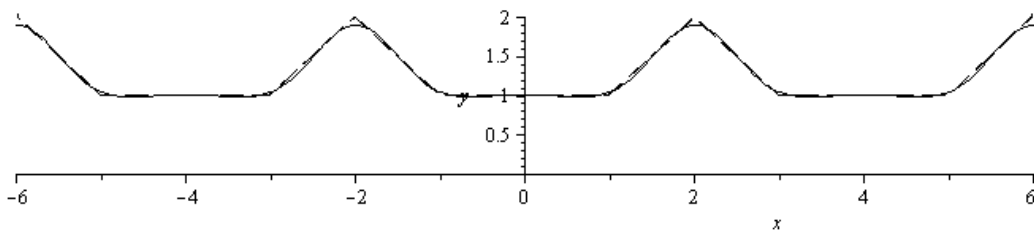
or

$$\frac{5}{4} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - \cos(n\pi/2)}{n^2} \cos \frac{n\pi x}{2}$$

The first four terms in the series are:

$$\frac{5}{4} + \frac{4}{\pi^2} \left(\frac{-1}{1^2} \cos \frac{\pi x}{2} + \frac{2}{2^2} \cos \frac{2\pi x}{2} - \frac{1}{3^2} \cos \frac{3\pi x}{2} - \dots \right).$$

When plotted, these four terms give a very good representation of the function:



Tutorial Exercises 2

1. Consider the function defined over one period by $f(x) = \begin{cases} -1 & \text{when } -2 < x \leq -1 \\ x & \text{when } -1 < x \leq 1 \\ 1 & \text{when } 1 < x \leq 2 \end{cases}$.

(a) Sketch the graph of the function for the range $-6 < x \leq 6$.

(b) Show that the Fourier series is $\sum_{n=1}^{\infty} \left(\frac{4 \sin(n\pi/2)}{n^2 \pi^2} - \frac{2(-1)^n}{n\pi} \right) \sin \left(\frac{n\pi x}{2} \right)$.

(c) Write down the first three terms of the series.

(d) State the value the series converges to when $x=2$.

2. The response of an electro-mechanical controller is modelled over one period by

$$f(x) = \begin{cases} 2 & \text{when } -4 < x \leq -2 \\ -x & \text{when } -2 < x \leq 0 \\ x & \text{when } 0 < x \leq 2 \\ 2 & \text{when } 2 < x \leq 4 \end{cases}.$$

(a) Sketch the function in the range $-12 < x \leq 12$.

(c) Show that the Fourier series of the function is

$$\frac{3}{2} + \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(n\pi/2) - 1}{n^2} \cos\left(\frac{n\pi x}{4}\right).$$

(c) Write down the first four terms of the series.

Homework Exercises 2

1. The response of an electro-mechanical controller is modelled over one period by

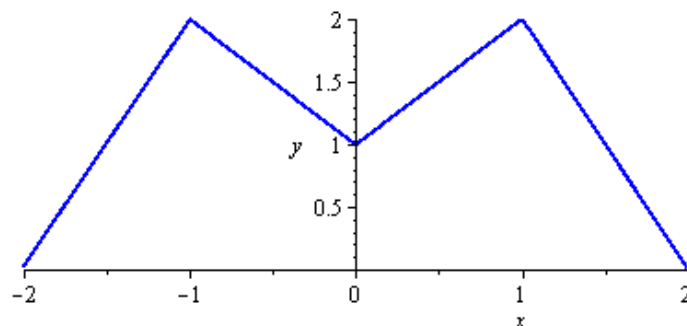
$$f(x) = \begin{cases} x & \text{when } -2 < x \leq -1 \\ -1 & \text{when } -1 < x \leq 0 \\ 1 & \text{when } 0 < x \leq 1 \\ x & \text{when } 1 < x \leq 2 \end{cases}$$

(a) Sketch the function in the range $-12 < x \leq 12$.

(b) Find the Fourier series of the function.

(c) Write down the first four terms of the series.

2. Find the Fourier series of the following function of period 4 and give the first three terms.



5. Complex Form of the Fourier Series

The Fourier series is sometime written in complex form to simplify calculations.

Consider the series of period 2π on the interval $[-\pi, \pi]$.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)].$$

The Euler formula states that $e^{i\theta} = \cos \theta + i \sin \theta$ and hence $e^{-i\theta} = \cos \theta - i \sin \theta$. Adding and subtracting gives the exponential forms of the cosine and sine functions:

$$\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \quad \text{and} \quad \sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}).$$

Hence we can write

$$\begin{aligned} a_n \cos(nx) + b_n \sin(nx) &= \frac{1}{2}a_n(e^{inx} + e^{-inx}) + \frac{1}{2i}b_n(e^{inx} - e^{-inx}) \\ &= \frac{1}{2}(a_n - ib_n)e^{inx} + \frac{1}{2}(a_n + ib_n)e^{-inx} \end{aligned}$$

Writing $\frac{1}{2}a_0 = c_0$, $\frac{1}{2}(a_n - ib_n) = c_n$ and $\frac{1}{2}(a_n + ib_n) = k_n$, and substituting gives:

$$f(x) = c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + k_n e^{-inx})$$

The coefficients are given by

$$\begin{aligned} c_n &= \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)(\cos nx - i \sin nx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} dx \\ k_n &= \frac{1}{2}(a_n + ib_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)(\cos nx + i \sin nx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{inx} dx. \end{aligned}$$

These are combined into a single formula by writing $k_n = c_{-n}$ so that the final form is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

with

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx} dx, \quad n = 0, \pm 1, \pm 2, \dots$$

Similar reasoning shows that a function of general period T the complex Fourier series is

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{i2n\pi x/T} \\ c_n &= \frac{1}{T} \int_{-T/2}^{T/2} f(x)e^{-i2n\pi x/T} dx, \quad n = 0, \pm 1, \pm 2, \dots \end{aligned}$$

Odd and Even Functions

The coefficients of the complex series can be written as:

$$\begin{aligned} c_n &= \frac{1}{T} \int_{-T/2}^{T/2} f(x) \left(\cos\left(\frac{2n\pi x}{T}\right) + i \sin\left(\frac{2n\pi x}{T}\right) \right) dx \\ &= \frac{1}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2n\pi x}{T}\right) dx + \frac{i}{T} \int_{-T/2}^{T/2} f(x) \sin\left(\frac{2n\pi x}{T}\right) dx \end{aligned}$$

If $f(x)$ is an even function, the second of these integrals is zero so that the coefficients are real. When recovering the real form of the series $c_n = c_{-n}$ so that the sine harmonics cancel and the series contains only cosine terms as expected.

If $f(x)$ is an odd function, the first of the integrals is zero so that the coefficients are purely imaginary. When recovering the real form of the series $c_n = -c_{-n}$ so that the cosine harmonics cancel and the series contains only sine terms as expected.

Examples

1. Show that the complex Fourier series of the square wave of period 4 given by

$$f(x) = \begin{cases} 0 & \text{when } -2 < x < 0 \\ 1 & \text{when } 0 < x < 2. \end{cases}$$

is

$$f(x) = \frac{1}{2} - \frac{i}{2\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1 - (-1)^n}{n} e^{in\pi x/2}.$$

Convert the series to real form.

2. Show that the complex Fourier series of the function of period 4 given by

$$f(x) = x \text{ for } 0 \leq x < 4 \text{ is}$$

$$f(x) = 2 + \frac{2i}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} e^{in\pi x/2}$$

Convert the series to real form.

Tutorial Exercises 3

1. Show that the complex Fourier series of the function defined over one period by

$$f(x) = \begin{cases} +2 & \text{for } -0.25 \leq x < 0 \\ -1 & \text{for } 0 \leq x < 0.25 \end{cases}$$

is

$$\frac{1}{2} + \frac{3i}{2\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1 - (-1)^n}{n} e^{i4n\pi x}.$$

Convert the series to real form and check your answer with Tutorial 1 Q1.

2. Show that the complex Fourier series of the function defined over one period by

$$f(x) = \begin{cases} 0 & \text{when } -1 < x \leq 0 \\ -x & \text{when } 0 \leq x < 1 \end{cases}$$

is

$$-\frac{1}{4} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\frac{-i(-1)^n}{2n\pi} + \frac{1 - (-1)^n}{2n^2\pi^2} \right] e^{in\pi x}.$$

Convert the series to real form and check your answer with Tutorial 1 Q2.

3. Show that the complex Fourier series of the function defined over one period by

$$f(x) = \begin{cases} -1 & \text{when } -2 < x \leq -1 \\ x & \text{when } -1 < x \leq 1 \\ 1 & \text{when } 1 < x \leq 2 \end{cases}.$$

is

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left[\frac{2i}{n^2\pi^2} \sin \frac{n\pi}{2} - \frac{i(-1)^n}{n\pi} \right] e^{in\pi x/2}.$$

Convert the series to real form and check your answer with Tutorial 2 Q1.

4. Find the complex Fourier series of the function defined over one period by

$$f(x) = \begin{cases} 2 & \text{when } -4 < x \leq -2 \\ -x & \text{when } -2 < x \leq 0 \\ x & \text{when } 0 < x \leq 2 \\ 2 & \text{when } 2 < x \leq 4 \end{cases}.$$

is

$$\frac{3}{2} + \frac{4}{\pi^2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\cos(n\pi/2) - 1}{n^2} e^{in\pi x/4}$$

Convert the series to real form and check your answer with Tutorial 2 Q2.