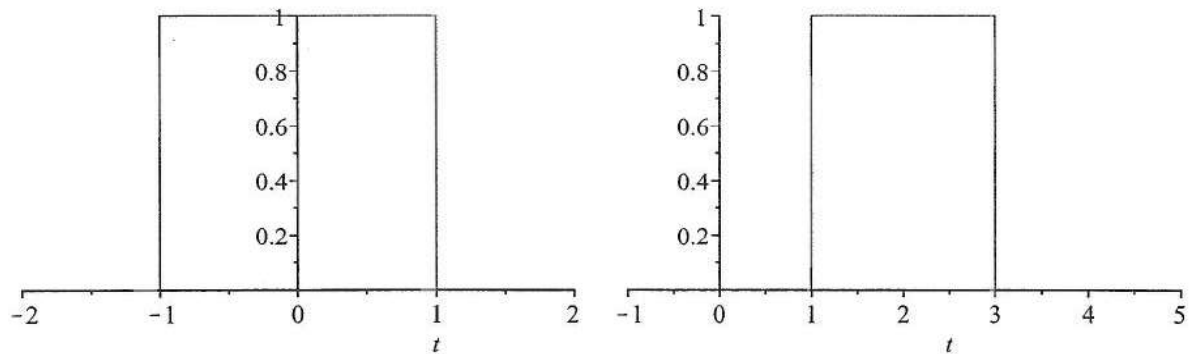


Properties of Fourier Transforms

We assume that $F(\omega)$, the Fourier Transform of $f(t)$ is known.

1. The Delay Property

Consider the function $f(t) = u(t+1) - u(t-1)$ and the function $f(t-2)$. Graphs of these two functions are shown below:



We can clearly see from the graphs that $f(t-2)$ is $f(t)$ shifted or **delayed** by 2 units. In general $f(t-b)$ is a delay of b units. The Fourier Transforms of $f(t-b)$ in terms of $F(\omega)$ is:

$$\text{FT } (f(t-b)) = e^{-i\omega b} F(\omega).$$

Proof

$$\begin{aligned} \text{Def: } F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\ \text{FT } \{f(t-b)\} &= \int_{-\infty}^{\infty} f(t-b) e^{-i\omega t} dt \\ u &= t-b \\ &= \int_{-\infty}^{\infty} f(u) e^{-i\omega(u+b)} du \\ &= e^{-i\omega b} \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du = e^{-i\omega b} F(\omega) \end{aligned}$$

Example 1

Use tables to find the Fourier Transform of $e^{-|t|}$ and hence find the Fourier Transform of $e^{-|t-4|}$.

$$\begin{aligned} \text{Formula book Page 7. No 6} \quad e^{-a|t|} &\rightarrow \frac{2a}{a^2 + \omega^2} \\ \therefore e^{-|t|} &\rightarrow \frac{2}{1 + \omega^2} \end{aligned}$$

$$\begin{aligned} f(t) &= e^{-|t|} \quad e^{-|t-4|} = f(t-4) \\ \therefore \text{FT } \{f(t-4)\} &= e^{-i4\omega} F(\omega) \\ &= e^{-i4\omega} \cdot \frac{2}{1 + \omega^2} \end{aligned}$$

2. Modulation

$$\text{FT} (e^{ibt} f(t)) = F(\omega - b).$$

Proof - an exercise for you to do

Example 2

Use tables to find the Fourier Transform of $u(t+a) - u(t-a)$ and hence find the Fourier Transform of $e^{2it}(u(t+a) - u(t-a))$.

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$$u(t+a) - u(t-a) = \begin{cases} 1 & \text{if } |t| < a \\ 0 & \text{otherwise} \end{cases} = g_{2a}(t) \quad T=2a$$

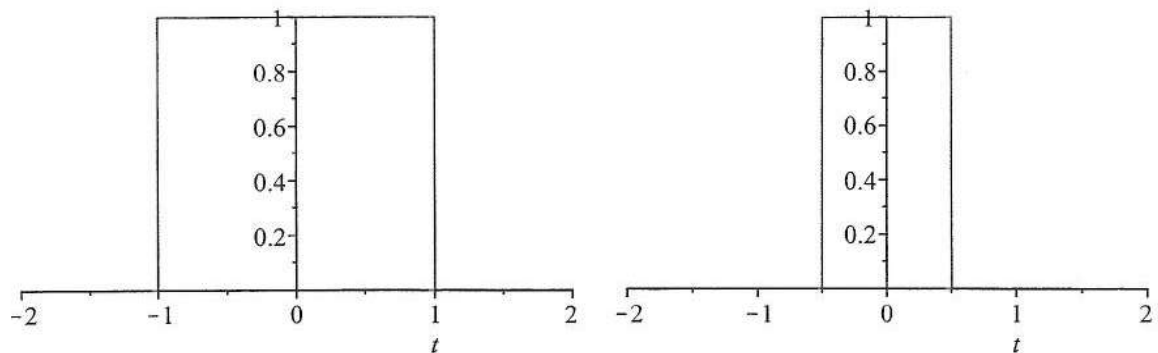
$$\begin{aligned} \text{FT} \{g_{2a}(t)\} &= T \text{sinc}\left(\frac{\omega T}{2}\right) \\ &= 2a \text{sinc}(\omega a) \end{aligned}$$

$$e^{2it} [u(t+a) - u(t-a)] = e^{2it} f(t)$$

$$\begin{aligned} \text{FT} \{e^{2it} f(t)\} &= F(\omega - 2) \\ &= 2a \text{sinc}((\omega - 2)a) \end{aligned}$$

3. Scaling Property

Again consider the function $f(t) = u(t+1) - u(t-1)$ and the function $f(2t)$. Graphs of these two functions are shown below:



We can clearly see from the graphs that $f(2t)$ is half the width of $f(t)$. In general $f(at)$ is $\frac{1}{a}$ -th the width of $f(t)$. Notice that the height is **not** changed. The relationship between the Fourier Transforms of $f(t)$ and $f(at)$ is:

$$\text{FT } (f(at)) = \frac{1}{a} F\left(\frac{\omega}{a}\right).$$

Proof

$$\begin{aligned} \text{FT } \{f(at)\} &= \int_{-\infty}^{\infty} f(at) e^{-i\omega t} dt \\ u=at & \\ &= \int_{-\infty}^{\infty} f(u) e^{-i\omega u/a} \cdot \frac{du}{a} \\ &= \frac{1}{a} \int_{-\infty}^{\infty} f(u) e^{-i\left(\frac{\omega}{a}\right)u} du \\ &= \frac{1}{a} F\left(\frac{\omega}{a}\right) \end{aligned}$$

Example 3

Use tables to find the Fourier Transform of $e^{-|t|}$ and hence find the Fourier Transform of $e^{-4|t|}$.

$$\begin{aligned} \text{From example 1: } f(t) &= e^{-|t|} & F(\omega) &= \frac{2}{1+\omega^2} \\ e^{-4|t|} &= f(4t) \\ \text{FT } \{f(4t)\} &= \frac{1}{4} F\left(\frac{\omega}{4}\right) \\ &= \frac{1}{4} \cdot \frac{2}{1+(\omega/4)^2} = \frac{8}{16+\omega^2}. \end{aligned}$$

Note

It is possible to take combinations of these properties.

Example 4

If $f(t) = u(t+1) - u(t-1)$ determine $F(\omega)$ and hence find the Fourier Transform of $f(2t-3)$.

$$f(t) = u(t+1) - u(t-1)$$

from example 2, with $a=1$

$$F(\omega) = 2 \operatorname{sinc}(\omega)$$

$$\text{FT } \{ f(2t-3) \} = \text{FT } \{ f(2(t-3/2)) \}$$

delay $t_0 = 3/2$

$$= e^{-3i\omega/2} \text{FT } \{ f(2t) \}$$

$$= e^{-3i\omega/2} \frac{1}{2} F(\omega/2) \quad \text{using scaling}$$

$$= \frac{1}{2} e^{-3i\omega/2} \cdot 2 \operatorname{sinc}(\omega/2)$$

$$= e^{-3i\omega/2} \operatorname{sinc}(\omega/2)$$

4. Time Differentiation Property

$$\text{FT} \left(\frac{df}{dt} \right) = i\omega F(\omega).$$

More generally:

$$\text{FT} \left(\frac{d^n f}{dt^n} \right) = (i\omega)^n F(\omega).$$

Proof - an exercise for you to do

Example 5

Use tables to find the Fourier Transform of $f(t) = e^{-t^2}$. Hence find the Fourier Transform of te^{-t^2} .

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$$e^{-at^2} \longleftrightarrow \sqrt{\frac{\pi}{a}} e^{-\omega^2/4a}$$

$$a=1 \quad e^{-t^2} \longleftrightarrow \sqrt{\pi} e^{-\omega^2/4}$$

$$\text{If } f(t) = e^{-t^2} \quad \frac{df}{dt} = -2te^{-t^2}$$

$$\text{F.T.} \left\{ \frac{df}{dt} \right\} = i\omega F(\omega) = i\omega \sqrt{\pi} e^{-\omega^2/4}$$

$$\text{FT} \left\{ -2te^{-t^2} \right\} = i\omega \sqrt{\pi} e^{-\omega^2/4}$$

$$\text{FT} \left\{ te^{-t^2} \right\} = -\frac{i\omega \sqrt{\pi}}{2} e^{-\omega^2/4}$$

5. Multiplication by t

$$\text{FT } (tf(t)) = i \frac{dF}{d\omega}$$

Proof

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$\begin{aligned} \frac{dF}{d\omega} &= \int_{-\infty}^{\infty} f(t) (-it) e^{-i\omega t} dt \\ &= -i \int_{-\infty}^{\infty} t f(t) e^{-i\omega t} dt \end{aligned}$$

$\times i$

$$i \frac{dF}{d\omega} = \int_{-\infty}^{\infty} t f(t) e^{-i\omega t} dt$$

Example 6

Use tables to find the Fourier Transform of $f(t) = e^{-t}u(t)$. Hence find the Fourier Transform of $te^{-t}u(t)$.

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$$e^{-at}u(t) \rightarrow \frac{1}{a+i\omega} \quad ; \quad e^{-t}u(t) \rightarrow \frac{1}{1+i\omega}$$

$$\begin{aligned} \text{FT } \{t e^{-t}u(t)\} &= i \frac{d}{d\omega} \left(\frac{1}{1+i\omega} \right) = i \frac{d}{d\omega} (1+i\omega)^{-1} \\ &= i (-1) (1+i\omega)^{-2} \cdot i \\ &= \frac{1}{(1+i\omega)^2} \end{aligned}$$

Example 7

Use tables to find the Fourier Transform of $f(t) = u(t+1) - u(t-1)$. Hence find the Fourier Transform of $t(u(t+1) - u(t-1))$.

From example 4 $F(\omega) = 2 \text{sinc}(\omega) = \frac{2 \sin(\omega)}{\omega}$

$$\text{FT } \{t [u(t+1) - u(t-1)]\} = i \frac{d}{d\omega} \cdot \frac{2 \sin(\omega)}{\omega}$$

$$= 2i \frac{d}{d\omega} (\sin(\omega) \omega^{-1})$$

$$= 2i \left\{ \cos(\omega) \omega^{-1} + \sin(\omega) (-1) \omega^{-2} \right\} \quad (\text{Product Rule})$$

$$= 2i \left\{ \frac{\cos(\omega)}{\omega} - \frac{\sin(\omega)}{\omega^2} \right\}$$

6. Symmetry Property

$$\text{FT } (F(t)) = 2\pi f(-\omega).$$

Proof - an exercise for you to do

Example 8

Use tables to find the Fourier Transform of $f(t) = e^{-2|t|}$. Hence find the Fourier Transform of $\frac{4}{4+t^2}$.

$$f(t) = e^{-2|t|} \Rightarrow F(\omega) = \frac{4}{4+\omega^2}$$

$$\begin{aligned} F(t) &= \frac{4}{4+t^2} & \text{FT } \{ F(t) \} &= 2\pi f(-\omega) \\ & & &= 2\pi e^{-2|-\omega|} \\ & & &= 2\pi e^{-2|\omega|} \end{aligned}$$

Example 9

Use tables to find the Fourier Transform of $f(t) = u(t+1) - u(t-1)$. Hence find the Fourier Transform of $\text{sinc}(t)$.

$$f(t) = u(t+1) - u(t-1) \Rightarrow F(\omega) = 2\text{sinc}(\omega)$$

$$\begin{aligned} F(t) &= 2\text{sinc}(t) & \text{FT } \{ F(t) \} &= 2\pi f(-\omega) \\ & & &= 2\pi [u(-\omega+1) - u(-\omega-1)] \end{aligned}$$

$$\therefore \text{FT } \{ \text{sinc}(t) \} = \pi [u(-\omega+1) - u(-\omega-1)]$$

Inverse Fourier Transforms

These properties can also be used in reverse

Example 10

Find the inverse Fourier Transforms of:

$$(a) \frac{5}{3+4iw} \quad (b) e^{-|w-3|} \quad (c) \frac{e^{4iw}}{9+w^2} \quad (d) \frac{6w}{9+w^2}$$

a) If $F(w) = \frac{5}{3+4iw} \Rightarrow f(t) = 5 e^{-3t} u(t)$

$$\frac{5}{3+4iw} = F(4w)$$

$$f(at) \rightarrow \frac{1}{|a|} F(w/a)$$

$$F\left(\frac{w}{a}\right) \Rightarrow |a| f(at)$$

$$F(4w) \Rightarrow a = 1/4$$

$$\therefore F(4w) \Rightarrow \frac{1}{4} f(t/4) = \frac{1}{4} 5 e^{-3t/4} u(t/4) \\ = \frac{5}{4} e^{-3t/4} u(t)$$

(b) $e^{-|w-3|} = F(w-3)$ where $F(w) = e^{-|w|}$

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$$\frac{1}{a^2+t^2} \Leftrightarrow \frac{\pi}{a} e^{-a|t|} \quad \therefore e^{-|t|} \Leftrightarrow \frac{1}{\pi(1+t^2)}$$

$$F(w-3) \rightarrow e^{3it} f(t) = \frac{e^{3it}}{\pi(1+t^2)}$$

(c) $\frac{e^{4iw}}{9+w^2} = e^{4iw} F(w) \rightarrow f(t+4)$ (delay)

If $F(w) = \frac{1}{9+w^2} = \frac{1}{6} \frac{2 \times 3}{3^2+w^2}$ then $f(t) = \frac{1}{6} e^{-3|t|}$

$$\therefore \frac{e^{4iw}}{9+w^2} \rightarrow \frac{1}{6} e^{-3|t+4|}$$

(d) $\frac{6w}{9+w^2} = \frac{1}{i} i w \cdot \frac{6}{9+w^2} = \frac{1}{i} i w F(w)$ where $F(w) = \frac{6}{9+w^2}$

$$\Rightarrow f(t) = e^{-3|t|}$$

$$i w F(w) \rightarrow \frac{df}{dt} \quad \therefore i w \frac{6}{9+w^2} \rightarrow \frac{d}{dt} e^{-3|t|} = -3 \operatorname{sgn}(t) e^{-3|t|}$$

$$\text{so } \frac{6w}{9+w^2} \Rightarrow \frac{1}{i} \cdot -3 \operatorname{sgn}(t) e^{-3|t|} = 3i \operatorname{sgn}(t) e^{-3|t|}$$