

Definitions

The Fourier Transform

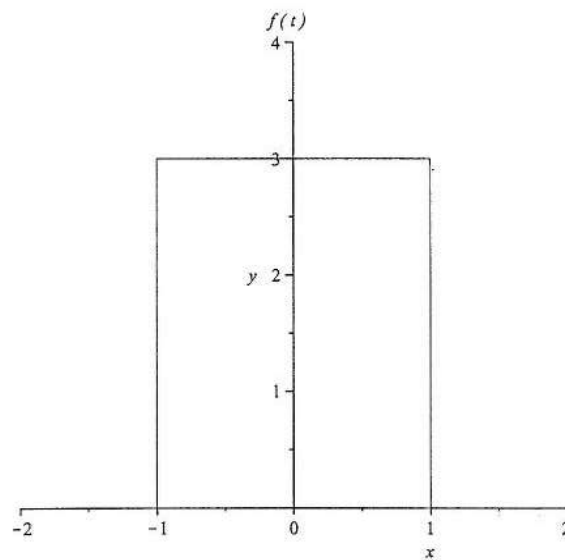
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt,$$

Inverse Fourier Transform

$$\text{FT}^{-1}(F(\omega)) = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega.$$

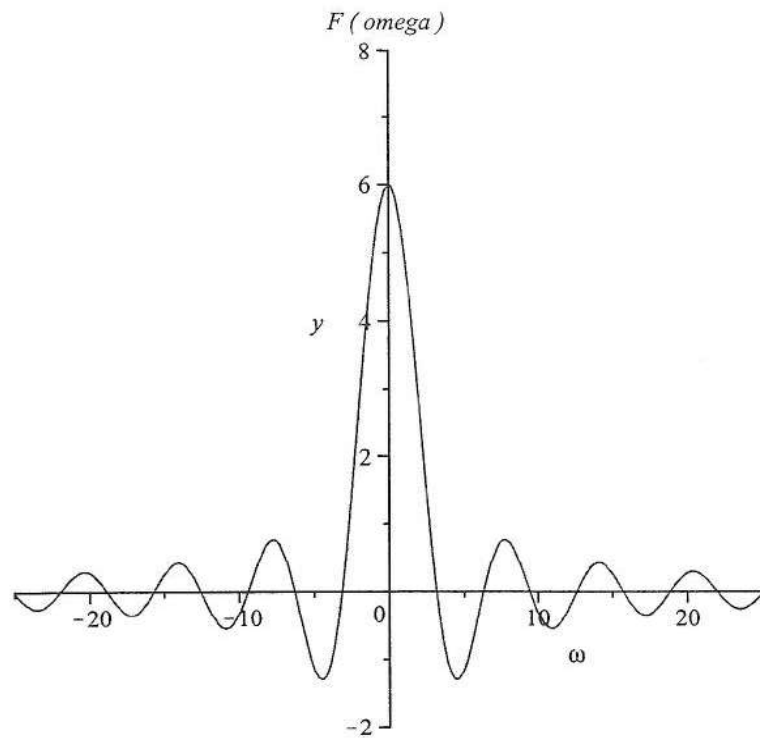
Example 1

Determine the Fourier Transforms of $f(t) = 3(u(t+1) - u(t-1))$



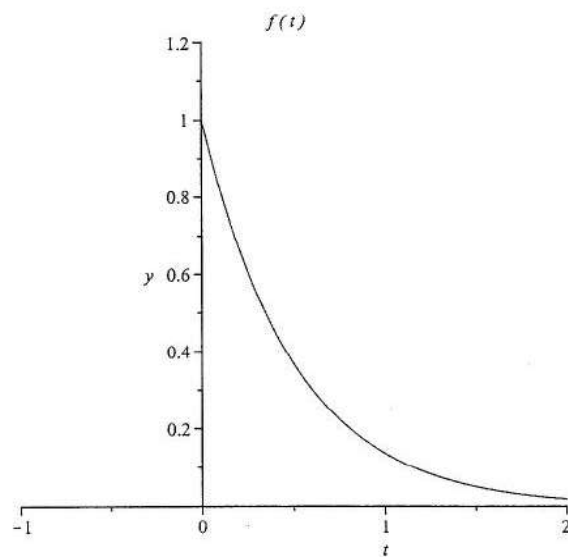
$$\begin{aligned} F(\omega) &= \int_{-1}^1 3 e^{-i\omega t} dt \\ &= \left[\frac{-3}{i\omega} e^{-i\omega t} \right]_{-1}^1 \\ &= \frac{3}{i\omega} [e^{i\omega} - e^{-i\omega}] \\ &= \frac{3}{i\omega} \cdot 2i \sin \omega = \frac{6 \sin \omega}{\omega} = 6 \operatorname{sinc}(\omega) \end{aligned}$$

Note: $F(\omega)$ is REAL in this example.



Example 2

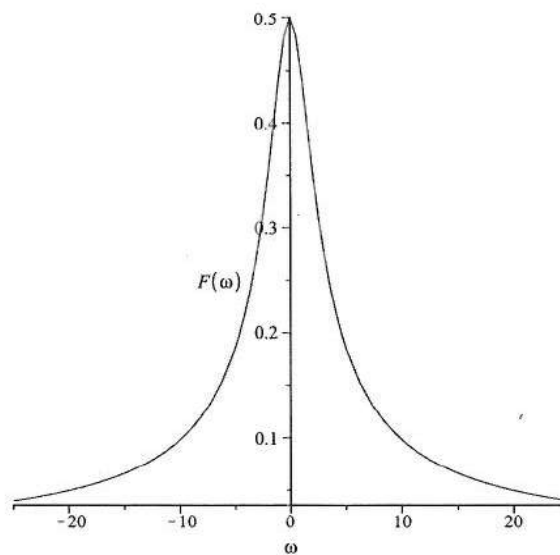
Determine the Fourier Transforms of $f(t) = u(t)e^{-2t}$



$$\begin{aligned}
 F(\omega) &= \int_0^{\infty} e^{-2t} \cdot e^{-i\omega t} dt \\
 &= \int_0^{\infty} e^{(-2-i\omega)t} dt \\
 &= \left[\frac{1}{-(2+i\omega)} e^{-(2+i\omega)t} \right]_0^{\infty} \\
 &= 0 - \frac{1}{-(2+i\omega)} \\
 &= \frac{1}{2+i\omega}
 \end{aligned}$$

Note: $F(\omega)$ is COMPLEX - draw the graph of the **modulus** of $F(\omega)$, which is known as the **amplitude spectra**.

$$\begin{aligned}
 |F(\omega)| &= \left| \frac{1}{1+2i\omega} \right| = \frac{1}{|2+i\omega|} \\
 &= \frac{1}{\sqrt{4+\omega^2}}
 \end{aligned}$$



Even Functions

If $f(t)$ is an **even** function then

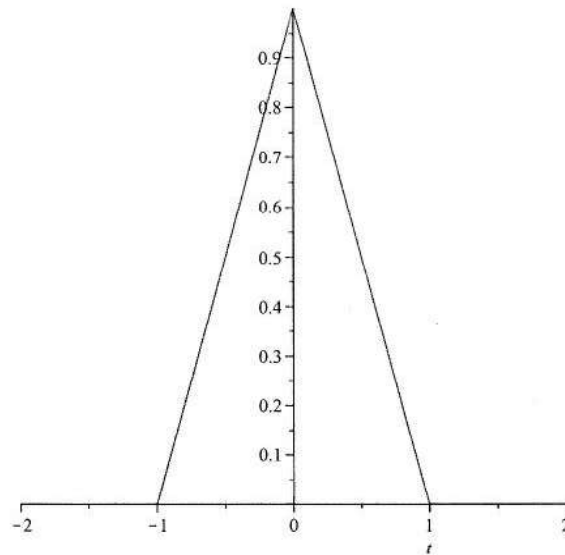
$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} f(t) \cos(\omega t) dt - i \int_{-\infty}^{\infty} f(t) \sin(\omega t) dt \\ &\quad \text{E} \times \text{E} = \text{E} \quad \text{E} \times \text{O} = \text{ODD} \rightarrow 0 \\ &= 2 \int_0^{\infty} f(t) \cos(\omega t) dt \end{aligned}$$

Note: The Fourier Transform of an EVEN function is REAL.

Example 3

Find the Fourier Transform of

$$f(t) = (1+t)(u(t+1) - u(t)) + (1-t)(u(t) - u(t-1))$$



$$F(\omega) = 2 \int_0^1 (1-t) \cos(\omega t) dt$$

$$= 2 \left(\left[(1-t) \frac{\sin(\omega t)}{\omega} \right]_0^1 - \int_0^1 (-1) \frac{\sin(\omega t)}{\omega} dt \right)$$

$$= 2 \left(0 - 0 + \frac{1}{\omega} \int_0^1 \sin(\omega t) dt \right)$$

$$= \frac{2}{\omega} \int_0^1 \sin(\omega t) dt$$

$$= \frac{2}{\omega} \left[-\frac{\cos(\omega t)}{\omega} \right]_0^1$$

$$= \frac{2}{\omega^2} [1 - \cos(\omega)]$$

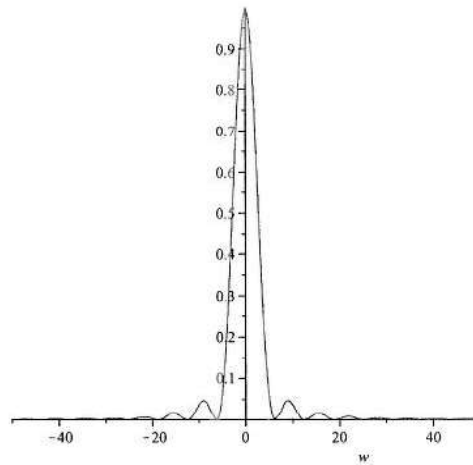
Note: $\cos 2A = 2\cos^2 A - 1$
 $= 1 - 2\sin^2 A$
 $= \cos^2 A - \sin^2 A$

$$\cos 0 = 1 - 2\sin^2(\omega/2)$$

$$\therefore F(\omega) = \frac{2}{\omega^2} \cdot 2\sin^2(\omega/2)$$

$$= \frac{4}{\omega^2} \sin^2(\omega/2)$$

$$= \frac{\sin^2(\omega/2)}{(\omega/2)^2} = \text{sinc}^2(\omega/2)$$



Odd Functions

If $f(t)$ is an odd function then

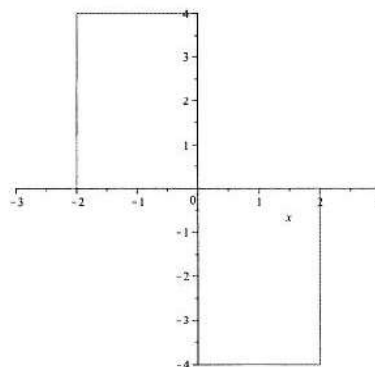
$$\begin{aligned}
 F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \\
 &= \int_{-\infty}^{\infty} \underbrace{f(t) \cos(\omega t)}_{0 \times E = 0} - i \int_{-\infty}^{\infty} \underbrace{f(t) \sin(\omega t)}_{0 \times O = E} dt \\
 &= -2i \int_0^{\infty} f(t) \sin(\omega t) dt
 \end{aligned}$$

Note: The Fourier Transform of an odd function is IMAGINARY.

Example 4

Find the Fourier Transform of

$$f(t) = 4(u(t+2) - u(t)) - 4(u(t) - u(t-2)).$$



$$F(\omega) = -2i \int_0^{\infty} f(t) \sin(\omega t) dt$$

$$= -2i \int_0^2 -4 \sin(\omega t) dt$$

$$= 8i \left[-\frac{\cos(\omega t)}{\omega} \right]_0^2$$

$$= -\frac{8i}{\omega} \{ \cos(2\omega) - 1 \}$$

$$\cos(2\omega) = 1 - 2\sin^2\omega$$

$$= 16i \frac{\sin^2\omega}{\omega}$$

