

1.

$$(a) f_n = n^2, \quad n = 0 \dots 3, \quad N = 4.$$

$$F_m = \sum_{n=0}^3 W_{mn} n^2,$$

$$W_{mn} = \exp\left(-\frac{2\pi i m n}{4}\right) = \exp\left(-i \frac{m n \pi}{2}\right).$$

$$W = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$$F_0 = \sum_{n=0}^3 W_{0n} n^2 = \sum_{n=0}^3 n^2 = 0 + 1 + 4 + 9 = 14$$

$$F_1 = \sum_{n=0}^3 W_{1n} n^2 = 0 - i - 4 + 9i = -4 + 8i$$

$$F_2 = \sum_{n=0}^3 W_{2n} n^2 = 0 - 1 + 4 - 9 = -6$$

$$F_3 = \sum_{n=0}^3 W_{3n} n^2 = 0 + i - 4 - 9i = -4 - 8i$$

$$(b) F_m = (-1)^m \quad n = 0 \dots 1, \quad N = 2$$

$$f_n = \frac{1}{2} \sum_{m=0}^1 W_{nm}^* (-1)^m$$

$$W_{nm}^* = \exp\left(+\frac{2\pi i n m}{2}\right) = \exp(+\pi i n m)$$

$$f_0 = \frac{1}{2} (1 - 1) = 0$$

$$f_1 = \frac{1}{2} (1 - \exp(\pi i)) = 1.$$

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(a) The remaining values of the DFT are

$$\begin{aligned}
 F_8 &= F_6^* = -2 + 3i, & F_9 &= F_5^* = 6 - 3i \\
 F_{10} &= F_4^* = -2 - 2i, & F_{11} &= F_3^* = 1 + 5i \\
 F_{12} &= F_2^* = 3 - 4i, & F_{13} &= F_1^* = -1 - 3i.
 \end{aligned}$$

$$(b) \quad f_n = \frac{1}{14} \sum_{m=0}^{13} F_m \exp(i m n \pi / 7)$$

$$\Rightarrow 14 f_0 = \sum_{m=0}^{13} F_m = F_0 + F_7 + 2 \operatorname{Re} \{ F_1 + F_2 + F_3 + F_4 + F_5 + F_6 \}$$

$$= 12 + 10 + 2(-1 + 3 + 1 - 2 + 6 - 2) = 22 + 10 = 32$$

$$f_0 = 32/14 = 16/7$$

(c) Similarly

$$14 f_7 = \sum_{m=0}^{13} F_m e^{i m \pi} = \sum_{m=0}^{13} F_m (-1)^m$$

$$\begin{aligned}
 &= F_0 - F_1 + F_2 - F_3 + F_4 - F_5 + F_6 - F_7 \\
 &\quad + F_8^* - F_9^* + F_{10}^* - F_{11}^* + F_{12}^* - F_{13}^*
 \end{aligned}$$

$$= (F_0 - F_7) - 2 \operatorname{Re} (F_1 + F_3 + F_5) + 2 \operatorname{Re} (F_2 + F_4 + F_6)$$

$$= (12 - 10) - 2(-1 + 1 + 6) + 2(3 - 2 - 2)$$

$$= 2 - 2(6) - (2) = -12. \quad f_7 = \frac{-12}{14} = \frac{-6}{7}$$

$$(d) \text{ From } F_m = \sum_{n=0}^{13} f_n \exp(-i \pi n m / 7)$$

$$\Rightarrow F_0 = \sum_{n=0}^{13} f_n = 12$$

$$(e) \text{ Similarly } F_{10} = \sum_{n=0}^{13} f_n \exp(-2 \pi i \frac{10n}{14})$$

$$= \sum_{n=0}^{13} f_n \exp(-\frac{\pi i}{7}(14-4)) = \sum_{n=0}^{13} f_n \exp(i \frac{4 \pi n}{7}) = -2 - 2i$$

$$2(f) \quad \text{Given} \quad F_m = \sum_{n=0}^{13} W_{nm} f_n$$

$$\text{then} \quad f_n = \frac{1}{14} \sum_{m=0}^{13} W_{nm}^* F_m$$

$$\text{but} \quad f_n^* = \frac{1}{14} \sum_{m=0}^{13} W_{nm} F_m^*$$

(Note $f_n^* = f_n$ since f_n is real but we do not need to use this).

$$|f_n|^2 = f_n^* f_n$$

$$\sum_{n=0}^{13} |f_n|^2 = \sum_{n=0}^{13} \left(\frac{1}{14} \right)^2 \sum_{m=0}^{13} \sum_{s=0}^{13} W_{nm}^* W_{ns} F_m F_s^*$$

$$\text{Now} \quad \sum_{n=0}^{13} W_{nm}^* W_{ns} = \sum_{n=0}^{13} \exp(-i \frac{\pi n}{7} (s-m))$$

Let $\theta = \frac{(s-m)\pi}{7}$ so that the sum is

$$1 + \exp(-i\theta) + \exp(-2i\theta) + \dots + \exp(-13i\theta) = \frac{1 - \exp(-14i\theta)}{1 - \exp(-i\theta)}$$

$$\exp(-14i\theta) = \exp(-2\pi i(s-m)) = 1$$

so that the sum is 1 unless $\exp(-i\theta) = 1$
i.e. $\theta = 0$ ($0 \leq m, s \leq 13$) $\Rightarrow m = s$.

In this case the sum is 14.

$$\sum_{n=0}^{13} |f_n|^2 = \left(\frac{1}{14} \right)^2 14 \sum_{m=0}^{13} |F_m|^2 = \frac{1}{14} \sum_{m=0}^{13} |F_m|^2$$

$$= \frac{1}{14} (F_0^2 + F_{13}^2 + 2 \sum_{n=1}^6 |F_n|^2) = \frac{1}{14} (144 + 100 + 2(10 + 25 + 26 + 8 + 45 + 13)) = \frac{1}{14} (244 + 254) = \frac{1}{14} (498) = \frac{249}{7} \approx 35.5$$

$$3 \quad F(\omega) = A \{ u(\omega+a) - u(\omega-a) \} \Rightarrow$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

$$= \frac{A}{2\pi} \int_{-a}^a e^{i\omega t} d\omega = \frac{A}{\pi} \int_0^a \cos \omega t d\omega$$

$$= \frac{A}{\pi} \left(\frac{\sin \omega t}{t} \right)_0^a = \frac{A}{\pi} \frac{\sin at}{t} = \frac{Aa}{\pi} \operatorname{sinc}(at)$$

$$\text{Thus } \mathcal{F}^{-1} \{ u(\omega+1) - u(\omega-1) \} = \frac{2}{\pi} \operatorname{sinc}(t)$$

$$\text{and } \mathcal{F}^{-1} \{ e^{-2i\omega\pi} F(\omega) \} = \frac{2}{\pi} \operatorname{sinc}(t-2\pi).$$

In case (a) we have

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \operatorname{sinc}(2\pi W t - n\pi)$$

where the bandwidth is $2\pi W = 1 \Rightarrow W = 1/2\pi$.

Thus $f(t)$ is in general an infinite sum

$$f(t) = \sum_{n=-\infty}^{\infty} f(n\pi) \operatorname{sinc}(t - n\pi) = \frac{2}{\pi} \operatorname{sinc}(t)$$

Thus in this case there is a single term when $n=0$ and $f(0) = 2/\pi$.

Similarly in case (b) $G(\omega) = F(\omega) + e^{-2i\omega\pi} F(\omega) \frac{1}{2}$ we have

$$\frac{2}{\pi} \operatorname{sinc}(t) + \frac{1}{\pi} \operatorname{sinc}(t-2\pi) \quad \text{and there are only}$$

two terms in the infinite sum $(f(0) = \frac{2}{\pi}, f(2) = \frac{1}{\pi})$.

$$1. N=8$$

$$W = \exp\left(-\frac{2\pi i}{8}\right) = \exp\left(-\frac{\pi i}{4}\right)$$

$$= \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} (1 - i)$$

$$W^0 = 1, \quad W = \frac{1}{\sqrt{2}} (1 - i)$$

$$W^2 = \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} = -i$$

$$W^3 = \cos \frac{3\pi}{4} - i \sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}} (-1 - i)$$

$$W^4 = \cos(-\pi) - i \sin(-\pi) = -1$$

$$W^5 = \cos \frac{5\pi}{4} - i \sin \frac{5\pi}{4} = \frac{1}{\sqrt{2}} (-1 + i) = (W^3)^*$$

$$W^6 = \cos \frac{3\pi}{2} - i \sin \frac{3\pi}{2} = i$$

$$W^7 = \cos \frac{7\pi}{4} - i \sin \frac{7\pi}{4} = \frac{1}{\sqrt{2}} (1 + i) = W^*$$

$$f(0) = f(000) = 0, \quad f(1) = f(001) = 6$$

$$f(2) = f(010) = 10, \quad f(3) = f(011) = 12$$

$$f(4) = 12 = f(100), \quad f(5) = f(101) = 10$$

$$f(6) = f(110) = 6, \quad f(7) = f(111) = 0$$

$$X(000) = f(000)W^0 + f(100)W^0 = 12$$

$$X(001) = f(000)W^0 + f(100)W^4 = 0 - 12 = -12$$

$$X(010) = f(001)W^0 + f(101)W^0 = 6 + 10 = 16$$

$$X(011) = f(001)W^0 + f(101)W^4 = 6 - 10 = -4$$

$$X(100) = f(010)W^0 + f(110)W^0 = 10$$

$$X(101) = f(010)W^0 + f(110)W^4 = 10 - 6 = 4$$

$$X(110) = f(011)W^0 + f(111)W^0 = 12$$

$$X(111) = f(011)W^0 + f(111)W^4 = 12$$

$$X_1(a_0 b_0 b_1) = X_0(0 a_0 b_0)W^0 + X_0(1 a_0 b_0)W^{4b_1 + 2b_0}$$

$$X_1(000) = X_0(000)W^0 + X_0(100)W^0 = 12 + 16 = 28$$

$$X_1(001) = X_0(000)W^0 + X_0(100)W^4 = 12 - 16 = -4$$

$$X_1(010) = X_0(001)W^0 + X_0(101)W^2 = -12 - 4i$$

$$X_1(011) = X_0(001)W^0 + X_0(101)W^6 = -12 + 4i$$

$$X_1(100) = X_0(010)W^0 + X_0(110)W^0 = 16 + 12 = 28$$

$$X_1(101) = X_0(010)W^0 + X_0(110)W^4 = 16 - 12 = 4$$

$$X_1(110) = X_0(011)W^0 + X_0(111)W^2 = -4 - 12i$$

$$X_1(111) = X_0(011)W^0 + X_0(111)W^6 = -4 + 12i$$

If we define, in the same pattern,

$$X_2(b_0 b_1 b_2) = X_1(0 b_0 b_1)W^0 + X_1(1 b_0 b_1)W^{4b_2 + 2b_1}$$

then $F(b_2 b_1 b_0) = X_2(b_0 b_1 b_2)$ (bit reversed)

$$F(000) = X_2(000) = X_1(000)W^0 + X_1(100)W^0 = 28 + 28 = 56$$

$$X_2(001) = X_1(000)W^0 + X_1(100)W^4 = 28 - 28 = 0 \\ = F(100)$$

$$X_2(010) = X_1(001)W^0 + X_1(101)W^2 = -4 - 4i = F(010)$$

$$X_2(011) = X_1(001)W^0 + X_1(101)W^6 = -4 + 4i = F(110)$$

$$X_2(100) = X_1(010)W^0 + X_1(110)W^0 = F(001) \\ = (-12 - 4i) + (-4 - 12i)W = -23.3137 - 9.656854i$$

$$X_2(101) = X_1(010)W^0 + X_1(110)W^5 = F(101) \\ = (-12 - 4i) + (-4 - 12i)W^5 = -0.686291 + 1.656854i$$

$$X_2(110) = X_1(011)W^0 + X_1(111)W^3 = F(011) = (F(101))^*$$

$$X_2(111) = X_1(011)W^0 + X_1(111)W^7 = F(111) = (F(001))^*$$

2. We may code $f_j = f(a, b)$

$$j = (a, b) \quad j = 2a + b$$

$$a = 0, 1, 2 \quad b = 0, 1$$

so that $\{0, 1, 2, 3, 4, 5\}$ are coded uniquely

The transform F_k may be coded as

$$F_k = F(c, d), \quad k \equiv (c, d), \quad k = 3c + d$$

$$c = 0, 1, \quad d = 0, 1, 2$$

so that $\{0, 1, 2, 3, 4, 5\}$ is coded uniquely

$$F(c, d) = \sum_{a=0}^2 \sum_{b=0}^1 f(a, b) W^{(2a+b)(3c+d)}$$

$$\text{since } F_k = \sum_j f_j W^{jk}, \quad W = \exp(-2\pi i/6).$$

$$F(c, d) = \sum_{b=0}^1 W^{b(3c+d)} \sum_{a=0}^2 f(a, b) W^{6ac+2ad}$$

Now $W^{6ac} = 1$ so we may define

$$X(b, d) = \sum_{a=0}^2 f(a, b) W^{2ad} \quad (\text{step 1})$$

and then

$$F(c, d) = \sum_{b=0}^1 X(b, d) W^{b(3c+d)}.$$

(Note in question the ranges of (a, b) and (c, d) are mistyped:

$$\begin{array}{ll} a = 0, 1, 2 & b = 0, 1 \\ c = 0, 1 & d = 0, 1, 2 \end{array}.)$$