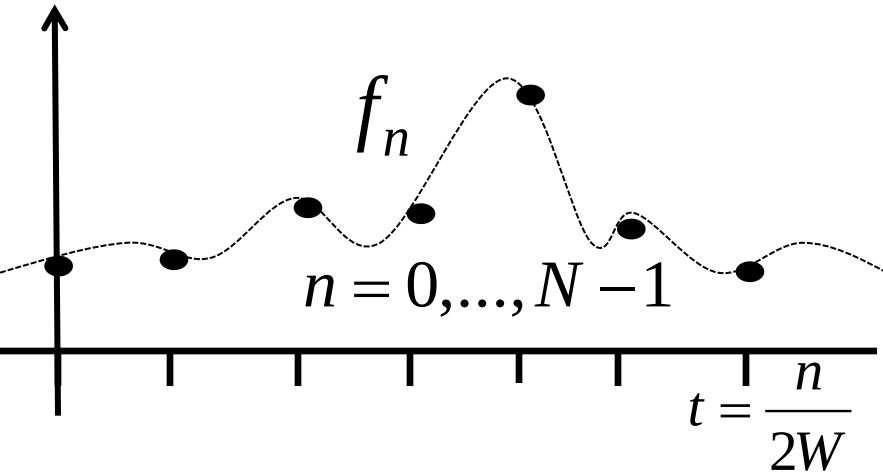


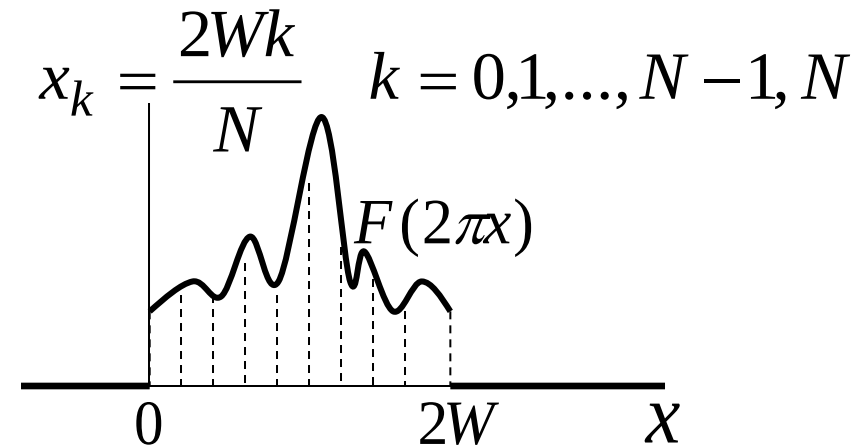
# Discrete Fourier Transforms



$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{i2\pi nk/N}$$

$(n = 0, \dots, N-1)$

Inverse Discrete  
Fourier Transform



$$F_m = \sum_{n=0}^{N-1} f_n e^{-i2\pi nm/N}$$

$(m = 0, \dots, N-1)$

Discrete Fourier  
Transform

Example 1. (a) Find DFT of  $f_n = n^2$ ,  $N=4$

$$f_n = 0, 1, 4, 9 \quad \Rightarrow \quad F_m = \sum_{n=0}^{N-1} f_n e^{-i2\pi mn/N}, \quad m = 0, 1, 2, 3$$

Example 1. (a) Find DFT of  $f_n = n^2$ ,  $N=4$

$$f_n = 0, 1, 4, 9 \Rightarrow F_m = \sum_{n=0}^3 f_n e^{-i2\pi nm/4}, \quad m = 0, 1, 2, 3$$

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$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} e^0 & e^0 & e^0 & e^0 \\ e^0 & e^{-\frac{i\pi}{2}} & e^{-i\pi} & e^{-\frac{3i\pi}{2}} \\ e^0 & e^{-i\pi} & e^{-2i\pi} & e^{-3i\pi} \\ e^0 & e^{-\frac{3i\pi}{2}} & e^{-3i\pi} & e^{-\frac{9i\pi}{2}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

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$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix}$$

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$$\begin{bmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4 + 8i \\ -6 \\ -4 - 8i \end{bmatrix} \quad \begin{matrix} \swarrow \\ \searrow \end{matrix} \quad F_3 = F_1^*$$

Example 2:  $F$  is the 14-point DFT of a real signal  $f$  of length 14.

We know that...

$$F[0] = 12$$

$$F[4] = -2 + 2i$$

$$F[1] = -1 + 3i$$

$$F[5] = 6 + 3i$$

$$F[2] = 3 + 4i$$

$$F[6] = -2 - 3i$$

$$F[3] = 1 - 5i$$

$$F[7] = 10$$

- (a) Evaluate the other components  
i.e.  $F[8], F[9], F[10], F[11], F[12], F[13]$

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$$\begin{bmatrix} F_0 \\ F_m \\ F_{14-m} \\ F_{13} \end{bmatrix} = \begin{bmatrix} W_{mn} \\ W_{(14-m)n} = W_{mn}^* \end{bmatrix} \begin{bmatrix} f_0 \\ f_n \\ f_{14-n} \\ f_{13} \end{bmatrix}$$

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(a) Evaluate the other components

i.e.  $F[8], F[9], F[10], F[11], F[12], F[13]$

$$F[0] = 12$$

$$F[1] = -1 + 3i$$

$$F[2] = 3 + 4i$$

$$F[3] = 1 - 5i$$

$$F[4] = -2 + 2i$$

$$F[5] = 6 + 3i$$

$$F[6] = -2 - 3i$$

$$F[7] = 10$$

$$F[14 - m] = F[m]^*$$

$$F[8] = F[6]^* = -2 + 3i$$

$$F[9] = F[5]^* = 6 - 3i$$

$$F[10] = F[4]^* = -2 - 2i$$

$$F[11] = F[3]^* = 1 + 5i$$

$$F[12] = F[2]^* = 3 - 4i$$

$$F[13] = F[1]^* = -1 - 3i$$

(b) Evaluate  $f[0]$

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{i2\pi nk/N} \Rightarrow f_n = \frac{1}{14} \sum_{k=0}^{13} F_k e^{i2\pi nk/14}$$

$$\Rightarrow f_0 = \frac{1}{14} \sum_{k=0}^{13} F_k e^0 = \frac{1}{14} \sum_{k=0}^{13} F_k$$

$$\begin{aligned} \Rightarrow 14 f_0 &= F_0 + F_1 + F_2 + F_3 + F_4 + F_5 + F_6 \\ &\quad + F_7 + F_8 + F_9 + F_{10} + F_{11} + F_{12} + F_{13} \\ &= F_0 + F_7 + 2\operatorname{Re}\{F_1 + F_2 + F_3 + F_4 + F_5 + F_6\} \\ &= 12 + 10 + 2\{-1 + 3 + 1 - 2 + 6 - 2\} \\ &= 32 \end{aligned}$$

$$\Rightarrow f_0 = 32/14 = 16/7$$

(b) Evaluate  $f[0]$

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{i2\pi nk/N} \Rightarrow f_n = \frac{1}{14} \sum_{k=0}^{13} F_k e^{i2\pi nk/14}$$

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$$\Rightarrow f_0 = 32/14 = 16/7$$

(c) Evaluate  $f[7]$

$$f_n = \frac{1}{N} \sum_{k=0}^{N-1} F_k e^{i2\pi nk/N} \Rightarrow f_n = \frac{1}{14} \sum_{k=0}^{13} F_k e^{i2\pi nk/14}$$

$$\Rightarrow f_7 = \frac{1}{14} \sum_{k=0}^{13} F_k e^{ik\pi} = \frac{1}{14} \sum_{k=0}^{13} F_k (-1)^k$$

$$\begin{aligned} \Rightarrow 14 f_7 &= F_0 - F_1 + F_2 - F_3 + F_4 - F_5 + F_6 \\ &\quad - F_7 + F_8 - F_9 + F_{10} - F_{11} + F_{12} - F_{13} \\ &= F_0 - F_7 + 2 \operatorname{Re}\{-F_1 + F_2 - F_3 + F_4 - F_5 + F_6\} \\ &= 12 - 10 + 2\{1 + 3 - 1 - 2 - 6 - 2\} \\ &= -12 \end{aligned}$$

$$\Rightarrow f_7 = -12/14 = -6/7$$

(d) Evaluate  $\sum_{n=1}^{13} f[n]$

$$F_m = \sum_{n=0}^{N-1} f_n e^{-i2\pi mn/N} \Rightarrow F_0 = \sum_{n=0}^{13} f_n e^0$$

$$\Rightarrow \sum_{n=0}^{13} f_n = F_0 = 12$$

(e) Evaluate  $\sum_{n=0}^{13} f[n]e^{4\pi in/7}$

$$F_m = \sum_{n=0}^{N-1} f_n e^{-i2\pi mn/N}$$

$$\sum_{n=0}^{13} f[n]e^{4\pi in/7} = \sum_{n=0}^{13} f_n e^{i2\pi n2/7} = \sum_{n=0}^{13} f_n e^{-i2\pi n5/7}$$

$$= \sum_{n=0}^{13} f_n e^{-i2\pi n10/14}$$

$$= F_{10} = -2 - 2i$$

(f) Evaluate  $\sum_{n=0}^{13} |f[n]|^2$

Since  $F_m = \sum_{n=0}^{13} f_n e^{-i2\pi nm/14} = \sum_{n=0}^{13} W_{mn} f_n$

Then  $f_n = \frac{1}{14} \sum_{m=0}^{13} F_m e^{i2\pi nm/14} = \frac{1}{14} \sum_{m=0}^{13} W_{nm}^* F_m$

and  $f_n^* = \frac{1}{14} \sum_{m=0}^{13} F_m^* e^{-i2\pi nm/14} = \frac{1}{14} \sum_{m=0}^{13} W_{nm} F_m^*$

$$\begin{aligned} |f_n|^2 = f_n f_n^* &\Rightarrow \sum_{n=0}^{13} |f_n|^2 = \sum_{n=0}^{13} \left( \frac{1}{14} \sum_{m=0}^{13} W_{nm}^* F_m \frac{1}{14} \sum_{s=0}^{13} W_{ns} F_s^* \right) \\ &= \sum_{n=0}^{13} \left( \frac{1}{14^2} \sum_{m=0}^{13} \sum_{s=0}^{13} W_{nm}^* W_{ns} F_m F_s^* \right) \end{aligned}$$

$$\Rightarrow \sum_{n=0}^{13} |f_n|^2 = \sum_{n=0}^{13} \left( \frac{1}{14^2} \sum_{m=0}^{13} \sum_{s=0}^{13} W_{nm}^* W_{ns} F_m F_s^* \right)$$

What is  $\sum_{n=0}^{13} W_{nm}^* W_{ns}$  ?

$$\sum_{n=0}^{13} W_{nm}^* W_{ns} = \sum_{n=0}^{13} e^{-i2\pi n(s-m)/14} = \begin{cases} 0 & m \neq s \\ 14 & m = s \end{cases}$$

(see solutions  
on Blackboard  
for details)

$$\Rightarrow \sum_{n=0}^{13} |f_n|^2 = \frac{1}{14^2} \times 14 \times \sum_{m=0}^{13} |F_m|^2$$

$$= \frac{1}{14} \times \left( F_0^2 + F_7^2 + 2 \left( |F_1^2|^2 + |F_2^2|^2 + |F_3^2|^2 + |F_4^2|^2 + |F_5^2|^2 + |F_6^2|^2 \right) \right)$$

$$= \frac{498}{14} = 35.5555$$

# Example 3

Show that the inverse FT of  $A(u(\omega + a) - u(\omega - a))$

is  $\frac{Aa}{\pi} \text{sinc}(at)$ .

Hence find the inverse FTs of

$$(a) \quad F(\omega) = 2(u(\omega + 1) - u(\omega - 1))$$

$$(b) \quad e^{-2\omega i\pi} F(\omega)$$

For both of these show that the formula for a band-limited signal:

$$\sum_{n=-\infty}^{\infty} f\left(\frac{2}{2W}\right) \text{sinc}(2\pi Wt - n\pi)$$

can be written as a finite sum of terms.

Inverse FT of  $F(\omega) = A(u(\omega + a) - u(\omega - a))$

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega = \frac{A}{2\pi} \int_{-a}^a e^{i\omega t} d\omega = \frac{2A}{2\pi} \int_0^a \cos(\omega t) d\omega \\ &= \frac{A}{\pi} \left[ \frac{\sin(\omega t)}{t} \right]_0^a = \frac{A}{\pi} \left( \frac{\sin(at)}{t} - \frac{\sin(0t)}{t} \right) \\ &= \frac{Aa}{\pi} \frac{\sin(at)}{at} = \frac{Aa}{\pi} \text{sinc}(at). \end{aligned}$$

$$\begin{aligned} (a) \quad FT^{-1}\{2(u(\omega + 1) - u(\omega - 1))\} &= \frac{2}{\pi} \text{sinc}(t) \\ (b) \quad FT^{-1}\{e^{-2\omega i\pi} F(\omega)\} &= \frac{2}{\pi} \text{sinc}(t - 2\pi) \quad (\text{Delay}) \end{aligned}$$

For a band-width limited signal

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \text{sinc}(2\pi Wt - n\pi)$$

$$(a) \quad f(t) = \frac{2}{\pi} \text{sinc}(t) \quad \Rightarrow 2\pi W = 1, n = 0$$

i.e. one non-zero term when  $n=0$ :

$$f(0) = \frac{2}{\pi}$$

$$(b) \quad f(t) = \frac{2}{\pi} \text{sinc}(t - 2\pi) \quad \Rightarrow 2\pi W = 1, n = 2$$

i.e. one non-zero term when  $n=2$ :

$$f(2) = \frac{2}{\pi}$$