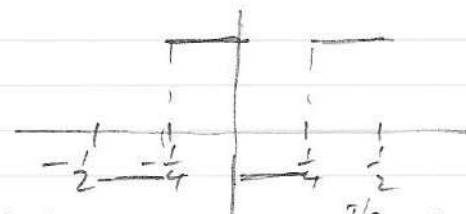


Tutorial Exercises 3

$$1) f(x) = \begin{cases} 2 & -1/4 \leq x < 0 \\ -1 & 0 \leq x \leq 1/4 \end{cases}$$



Complex series is $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i 2n\pi x}$, $c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-i n\pi x} dx$

$$c_0 = \frac{1}{1/2} \int_{-1/4}^{1/4} f(x) dx = 2 \left[\int_{-1/4}^0 2 dx + \int_0^{1/4} (-1) dx \right] = 2 \left[\left[2x \right]_{-1/4}^0 + \left[-x \right]_0^{1/4} \right]$$

$$= 2 \left[(0 - (-1/2)) + (-1/4 - (-0)) \right] = 2 \left[\frac{1}{2} - \frac{1}{4} \right] = \frac{1}{2}$$

$$c_n = \frac{1}{1/2} \int_{-1/4}^{1/4} f(x) e^{-i n\pi x} dx = 2 \left[\int_{-1/4}^0 2 e^{-i 4n\pi x} dx + \int_0^{1/4} (-1) e^{-i 4n\pi x} dx \right]$$

$$= 2 \left[-\frac{2}{i 4n\pi} e^{-i 4n\pi x} \Big|_{-1/4}^0 - \frac{1}{i 4n\pi} e^{-i 4n\pi x} \Big|_0^{1/4} \right]$$

$$= 2 \left[\frac{-2}{i 4n\pi} (1 - e^{+i n\pi}) + \frac{1}{i 4n\pi} (e^{-i n\pi} - 1) \right]$$

$$= 2 \left[\frac{-2}{i 4n\pi} (1 - (\cos n\pi + i \sin n\pi)) - \frac{1}{i 4n\pi} (1 - (\cos n\pi - i \sin n\pi)) \right]$$

$$= 2 \cdot \frac{-3(1 - (-1)^n)}{i 4n\pi} = \frac{-3(1 - (-1)^n)}{i 2n\pi}$$

i. Complex series is

$$\frac{1}{2} - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{3(1 - (-1)^n)}{i 2n\pi} e^{i 4n\pi x}$$

Recovering the real form:

$$c_n e^{i 4n\pi x} = \frac{-3(1 - (-1)^n)}{i 2n\pi} (\cos 4n\pi x + i \sin 4n\pi x)$$

$$c_{-n} e^{-i 4n\pi x} = \frac{-3(1 - (-1)^{-n})}{-i 2n\pi} (\cos 4n\pi x - i \sin 4n\pi x)$$

Summing gives:

$$\frac{1}{2} - \sum_{n=1}^{\infty} \frac{3(1 - (-1)^n)}{n\pi} \sin 4n\pi x$$

$$2. \quad f(x) = \begin{cases} 0 & -1 < x \leq 0 \\ -x & 0 \leq x < 1 \end{cases}$$



Complex series: $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i 2 n \pi x / T}$, $C_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-i 2 n \pi x / T} dx$

$$C_0 = \frac{1}{2} \int_{-1}^1 f(x) dx = \frac{1}{2} \left[\int_{-1}^0 0 dx + \int_0^1 (-x) dx \right] = \frac{1}{2} \left[-\frac{1}{2} x^2 \right]_0^1 = \frac{1}{2} \left(-\frac{1}{2} - 0 \right) = -\frac{1}{4}$$

$$C_n = \frac{1}{2} \int_{-1}^1 f(x) e^{-i n \pi x} dx = \frac{1}{2} \int_0^1 -x \cdot e^{-i n \pi x} dx$$

$$= \frac{1}{2} \left[-x \cdot \frac{1}{-i n \pi} e^{-i n \pi x} \Big|_0^1 - \int_0^1 (-1) \cdot \frac{1}{-i n \pi} e^{-i n \pi x} dx \right]$$

$$= \frac{1}{2} \left[\left(\frac{1}{i n \pi} e^{-i n \pi} - 0 \right) + \frac{1}{(i n \pi)^2} e^{-i n \pi x} \Big|_0^1 \right]$$

$$= \frac{1}{2} \left[\frac{1}{i n \pi} e^{-i n \pi} - \frac{1}{n^2 \pi^2} (e^{-i n \pi} - 1) \right]$$

$$= \frac{1}{2} \left[\frac{1}{i n \pi} (\cos(n\pi) - i \sin(n\pi)) + \frac{1}{n^2 \pi^2} (1 - (\cos(n\pi) - i \sin(n\pi))) \right]$$

$$= \frac{1}{2} \left[\frac{(-1)^n}{i n \pi} + \frac{1 - (-1)^n}{n^2 \pi^2} \right]$$

$\therefore$ Complex series is

$$-\frac{1}{4} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{2} \left[\frac{(-1)^n}{i n \pi} + \frac{1 - (-1)^n}{n^2 \pi^2} \right] e^{i n \pi x}$$

Recovering the real form:

$$C_n e^{i n \pi x} = \frac{1}{2} \left[\frac{(-1)^n}{i n \pi} + \frac{1 - (-1)^n}{n^2 \pi^2} \right] (\cos n \pi x + i \sin n \pi x)$$

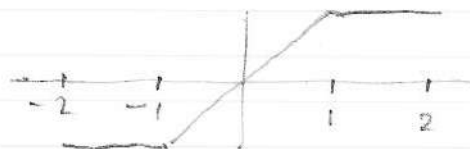
$$C_{-n} e^{-i n \pi x} = \frac{1}{2} \left[\frac{(-1)^{-n}}{-i n \pi} + \frac{1 - (-1)^{-n}}{n^2 \pi^2} \right] (\cos n \pi x - i \sin n \pi x)$$

Summing gives:

$$-\frac{1}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{n \pi} \sin n \pi x + \frac{1 - (-1)^n}{n^2 \pi^2} \cos n \pi x \right]$$

3.

$$f(x) = \begin{cases} -1 & -2 < x < -1 \\ x & -1 < x < 1 \\ 1 & 1 < x < 2 \end{cases}$$



Complex series: $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i 2n\pi x/T}$, $c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-i 2n\pi x/T} dx$

$$c_0 = \frac{1}{4} \int_{-2}^2 f(x) dx = 0 \text{ since } f \text{ is odd.}$$

$$c_n = \frac{1}{4} \int_{-2}^2 f(x) e^{-i 2n\pi x/4} dx = \frac{1}{4} \int_{-2}^2 f(x) \left(\cos \frac{n\pi x}{2} - i \sin \frac{n\pi x}{2} \right) dx$$

$$= \frac{1}{4} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx - \frac{1}{4} i \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx$$

$$= 0 - 2 \times \frac{1}{4} i \int_0^2 f(x) \sin \frac{n\pi x}{2} dx$$

$$= -\frac{i}{2} \left[\int_0^1 x \sin \frac{n\pi x}{2} dx + \int_1^2 1 \cdot \sin \frac{n\pi x}{2} dx \right]$$

$$= -\frac{i}{2} \left[\frac{4}{n^2 \pi^2} \sin \frac{n\pi}{2} - \frac{2(-1)^n}{n\pi} \right] \quad (\text{see solution to Tutorial 2, Q1})$$

$\therefore$ Complex series is

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{-i}{2} \left[\frac{4}{n^2 \pi^2} \sin \frac{n\pi}{2} - \frac{2(-1)^n}{n\pi} \right] e^{i n\pi x/2}$$

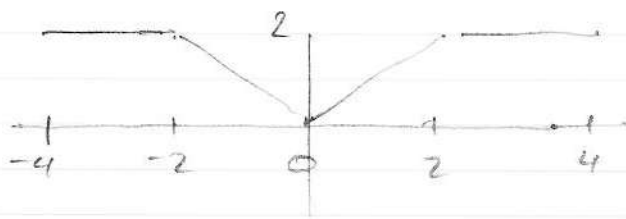
Recovering the real form:

$$c_n e^{i n\pi x/2} = -\frac{i}{2} \left[\frac{4}{n^2 \pi^2} \sin \frac{n\pi}{2} - \frac{2(-1)^n}{n\pi} \right] \left(\cos \frac{n\pi x}{2} + i \sin \frac{n\pi x}{2} \right)$$

$$c_{-n} e^{-i n\pi x/2} = -\frac{i}{2} \left[\frac{4}{(-n)^2 \pi^2} \sin \frac{-n\pi}{2} - \frac{2(-1)^{-n}}{-n\pi} \right] \left(\cos \frac{n\pi x}{2} - i \sin \frac{n\pi x}{2} \right)$$

Sum:
$$\sum_{n=1}^{\infty} \left[\frac{4}{n^2 \pi^2} \sin \frac{n\pi}{2} - \frac{2(-1)^n}{n\pi} \right] \sin \frac{n\pi x}{2}$$

4.
$$f(x) = \begin{cases} 2 & -4 \leq x \leq -2 \\ -x & -2 \leq x \leq 0 \\ x & 0 \leq x \leq 2 \\ 2 & 2 \leq x \leq 4 \end{cases}$$



Complex series: $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2n\pi x/T}$, $c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x) e^{-i2n\pi x/T} dx$

$c_0 = \frac{1}{8} \int_{-4}^4 f(x) dx = \frac{2}{8} \int_0^4 f(x) dx$ (since f is even)

$= \frac{1}{4} \left[\int_0^2 x dx + \int_2^4 2 dx \right] = \frac{1}{4} \left[\left[\frac{1}{2} x^2 \right]_0^2 + [2x]_2^4 \right]$

$= \frac{1}{4} \left[(2-0) + (8-4) \right] = \frac{3}{2}$

$c_n = \frac{1}{8} \int_{-4}^4 f(x) e^{-i2n\pi x/8} dx = \frac{1}{8} \int_{-4}^4 f(x) (\cos \frac{n\pi x}{4} - i \sin \frac{n\pi x}{4}) dx$

$= \frac{2}{8} \int_0^4 f(x) \cos \frac{n\pi x}{4} dx + 0$ (since f is even)

$= \frac{1}{4} \left[\int_0^2 x \cos \frac{n\pi x}{4} dx + \int_2^4 2 \cos \frac{n\pi x}{4} dx \right]$

$= \frac{4}{n^2 \pi^2} (\cos \frac{n\pi}{2} - 1)$ (see solution to Tutorial 2, Q2)

$\therefore$ Complex series is

$$\frac{3}{2} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{4}{n^2 \pi^2} (\cos \frac{n\pi}{2} - 1) e^{in\pi x/4}$$

Recovering the real form:

$c_n e^{in\pi x/4} = \frac{4}{n^2 \pi^2} (\cos \frac{n\pi}{2} - 1) (\cos \frac{n\pi x}{4} + i \sin \frac{n\pi x}{4})$

$c_{-n} e^{-in\pi x/4} = \frac{4}{(-n)^2 \pi^2} (\cos \frac{n\pi}{2} - 1) (\cos \frac{n\pi x}{4} - i \sin \frac{n\pi x}{4})$

Sum:
$$\frac{3}{2} + \sum_{n=1}^{\infty} \frac{8}{n^2 \pi^2} (\cos \frac{n\pi}{2} - 1) \cos \frac{n\pi x}{4}$$