PW #2: 2nd-order ordinary differential equations

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This work is devoted to the resolution of the 1D Poisson equation:

$$-u''(x) = f(x). \tag{1}$$

f is a given function over $[0,1]$. We shall consider two kinds of boundary conditions. First of all, we shall investigate the case of a Cauchy problem where (1) is supplemented with:

$$u(0) = 1, \qquad u'(0) = \alpha. \tag{2}$$

Then, we shall focus on the Dirichlet problem:

$$u(0) = 1, \qquad u(1) = 0. \tag{3}$$

For all the simulations, we set: $f(x) = e^x$. Given an integer N, the interval is discretized as $x_i = (i-1)\Delta x$, $1 \le i \le N+1$, $\Delta x = \frac{1}{N}$.

Exercise 1

Write out a program solving (1-2) by means of the finite difference method for any $\alpha \in \mathbb{R}$.

Exercise 2

We now aim at solving (1-3).

1. *Write out the linear system corresponding to the unknowns $(u_0, u_1, \dots, u_{N+1})$.*
2. *What does the system become when taking only $(u_1, \dots, u_N)$ into account?*
3. *Using the \ function of* MATLAB, *implement the resolution of the equation for the two previous cases. Do you notice any difference as N grows?*
4. *Plot the numerical solutions for various N.*

Exercise 3

We propose an alternative method for solving (1-3) called the shooting method. *It consists in finding the suitable α in* (2) *which matches the condition* (3). *More precisely, using the algorithm from* ***Exercise 1****, start with $\alpha_0 = 0$ and $\alpha_1 = -1$. For each value, compare the resulting u_{N+1} to the value prescribed by* (3). *Apply the bisection method to update α_n at each step.*

The bisection method for finding the zero of a function consists in dividing by 2 the interval in which we are looking for the solution. In this case, we are intending to find α such that $u_\alpha(1) = 0$. Thus given α_0 and $\alpha_1 > \alpha_0$ such that $u_{\alpha_0}(1) < 0$ and $u_{\alpha_1}(1) > 0$, set $I = [\alpha_0, \alpha_1]$ and $\alpha_2 = \frac{\alpha_0+\alpha_1}{2}$.

1. Compute $u_{\alpha_2}(1)$:
   (a) If $u_{\alpha_2}(1) < 0$, then $I = [\alpha_2, \alpha_1]$ and $\alpha_3 = \frac{\alpha_1+\alpha_2}{2}$;
   (b) If $u_{\alpha_2}(1) > 0$, then $I = [\alpha_0, \alpha_2]$ and $\alpha_3 = \frac{\alpha_0+\alpha_2}{2}$;
2. Iterate.

Exercise 4

We investigate in this last exercise an iterative method to solve the linear system $\mathbf{A}\bar{u} = \bar{f}$ derived in Exercise 2.2. Let D be the diagonal matrix with coefficients 2. Let E be the sparse matrix whose entries are $E_{i,i-1} = 1$. Then $\mathbf{A} = D - E - {}^tE$.

1. *The equality $\mathbf{A}\bar{u} = \bar{f}$ also reads $D\bar{u} = \bar{f} + E\bar{u} + {}^tE\bar{u}$. Hence the iterative Jacobi method:*

$$D\bar{u}^{n+1} = \bar{f} + E\bar{u}^n + {}^tE\bar{u}^n.$$

   Implement this method with different stopping criteria.

2. *The equality $\mathbf{A}\bar{u} = \bar{f}$ may also be written $(D+E)\bar{u} = \bar{f} + {}^tE\bar{u}$. Hence the iterative Gauss-Seidel method:*

$$(D+E)\bar{u}^{n+1} = \bar{f} + {}^tE\bar{u}^n.$$

   Implement this method and compare the number of iterations that this method required to reach the criterion to the one using Jacobi.